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MAGNETIC INDUCTION OF SPHERICAL AND PROLATE SPHEROIDAL BODIES WITH INFINITESIMALLY THIN CURRENT BANDS HAVING A COMMON AXIS OF SYMMETRY AND IN A UNIFORM INDUCING FIELD – A SUMMARY

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DAVID W. TAYLOR NAVAL SHIP RESEARCH AND DEVELOPMENT CENTER

Bethesda, Maryland 20084

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by

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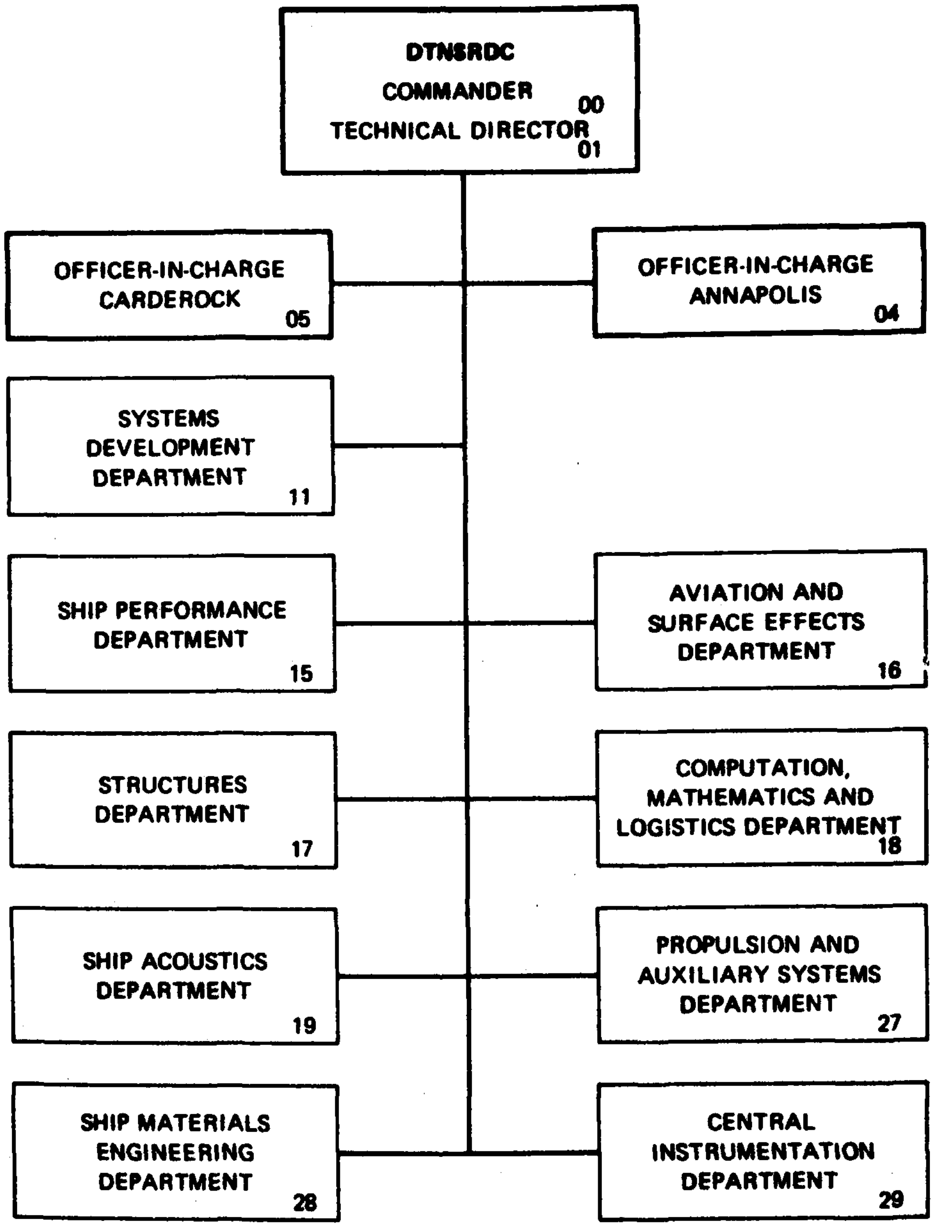

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Magnetic induction is calculated for several configurations of ferromagnetic spherical and prolate spheroidal bodies (hollow and solid) with internal and/or external infinitesimally thin spherical and spheroidal current bands, respectively. Magnetic induction is presented for ferromagnetic spherical and spheroidal bodies in a constant inducing field of arbitrary orientation. The ferromagnetic bodies are assumed to be linear and homogeneous. The reduction

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A

TABLE OF CONTENTS

LIST OF FIGURES

Page

NOMENCLATURE

$\overline{A}$	Vector Potential function
A_{ψ}	Psi (ψ) component of $\overline{A}$
$A_{\psi I}$	Psi component of $\overline{A}$ in region I
$A_{\psi II}$	Psi component of $\overline{A}$ in region II
$A_{\psi III}$	Psi component of $\overline{A}$ in region III
$A_{\psi IV}$	Psi component of $\overline{A}$ in region IV
$A_{\psi V}$	Psi component of $\overline{A}$ in region V
$\overline{B}$	Magnetic flux density or magnetic induction
$\overline{B}_1$	Magnetic flux density in medium I
$\overline{B}_2$	Magnetic flux density in medium II
B_{n1}	Normal component of $\overline{B}$ in medium 1
B_{n2}	Normal component of $\overline{B}$ in medium 2
B_r	Radial component of the magnetic flux density
B_{rI}	Radial component of $\overline{B}$ in region I
B_{rII}	Radial component of $\overline{B}$ in region II
B_{rIII}	Radial component of $\overline{B}$ in region III
B_{rIV}	Radial component of $\overline{B}$ in region IV
B_{rV}	Radial component of $\overline{B}$ in Region V
B_{η}	Eta component of the magnetic flux density
$B_{\eta I}$	Eta component of $\overline{B}$ in region I

$B_{\eta II}$	Eta component of $\overline{B}$ in region II
$B_{\eta III}$	Eta component of $\overline{B}$ in region III
$B_{\eta IV}$	Eta component of $\overline{B}$ in region IV
$B_{\eta V}$	Eta component of $\overline{B}$ in region V
B_{θ}	Theta component of the magnetic flux density
dv	Elemental volume
$\hat{e}_r$	Unit normal vector in radial direction
$\hat{e}_{\eta}$	Unit normal vector in eta direction
$\hat{e}_{\theta}$	Unit normal vector in theta direction
$\hat{e}_{\psi}$	Unit normal vector in azimuthal direction
e_1, e_2, e_3	Metric coefficients for a prolate spheroidal coordinate system
$\overline{H}$	Magnetic field intensity
$\overline{H}_o$	Externally applied uniform field
H_{t1}	Tangential component of $\overline{H}$ in medium 1
H_{t2}	Tangential component of $\overline{H}$ in medium 2
$\overline{H}_1$	Magnetic field intensity in medium 1
$\overline{H}_2$	Magnetic field intensity in medium 2
I	Electric current
J	Magnitude of $\overline{J}_s$
$\overline{J}$	Electric current density
J_r	Radial component of $\overline{J}$
$\overline{J}_s$	Surface current density

J_η	Eta component of $\overline{J}$
J_θ	Theta component of $\overline{J}$
J_ψ	Psi component of $\overline{J}$
$\overline{J}_1$	Electric current density of internal current band
$\overline{J}_2$	Electric current density of external current band
J_1	Magnitude of $\overline{J}_1$
J_2	Magnitude of $\overline{J}_2$
$\overline{n}_{12}$	Unit vector normal to interface, directed from medium 1 into medium 2
$P_p^m(\cos\theta)$	Associated Legendre function of the first kind
P_p^Δ	Variable used for simplification
p	Integer from one to infinity
$Q_p^m(\cos\theta)$	Associated Legendre function of the second kind
$R_i (i=1,2,3)$	Component of radius vector to boundary i which has spherical symmetry
r	Radius of spherical coordinate system
r,θ,ψ	Spherical coordinates
r'	Distance of the point where $\overline{A}$ is being determined from
x,y,z	Rectangular coordinates
$\eta_1,\eta_2,\eta_3,\eta_4$	Constants (specified values of η)
η,θ,ψ	Prolate spheroidal coordinates
μ	Magnetic permeability
μ_r	Relative magnetic permeability

μ_o	Permeability of free space
μ_1	Permeability of medium 1
μ_2	Permeability of medium 2
ξ	Variable equal to cosh η
ν	Variable equal to cos θ
ϕ_m	Magnetic scalar potential
χ_m	Magnetic susceptibility
✡	Vector Laplacian operator
✡ A_ψ	Psi vector component of the vector Laplacian of $\overline{A}$ in spherical or prolate spheroidal coordinates
✡ $\overline{A}_\psi$	Vector Laplacian of $\overline{A}$ in prolate spheroidal coordinates

$$= \frac{(\sinh^2 \eta + \sin^2 \theta)^{-1/2}}{a^2(\sinh \eta \sin \theta)} \hat{e}_\psi \left(\frac{\partial}{\partial \eta} \frac{1}{\sinh \eta} \frac{\partial}{\partial \eta} (\sinh \eta \, A_\psi) + \frac{\partial}{\partial \theta} \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (\sin \theta \, A_\psi) \right)$$

✡ $\overline{A}_\psi$	Vector Laplacian of $\overline{A}$ in spherical coordinates

$$= \hat{e}_\psi \left[\frac{\partial^2 A_\psi}{\partial r^2} + \frac{2}{r} \frac{\partial A_\psi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 A_\psi}{\partial \theta^2} + \frac{\cot \theta}{r^2} \frac{\partial A_\psi}{\partial \theta} - \frac{A_\psi}{r^2 \sin^2 \theta} \right]$$

∇^2	Scalar Laplacian operator
$\overline{\nabla} \cdot$	Divergence operator
$\overline{\nabla} \times$	Curl operator
$(\overline{\nabla} \times \overline{A})_r$	Radial component of the curl of $\overline{A}$

$(\overline{\nabla}\times\overline{A})_\eta$	Eta component of the curl $\overline{A}$
$(\overline{\nabla}\times\overline{A})_\theta$	Theta component of the curl $\overline{A}$

EXECUTIVE SUMMARY

OBJECTIVE

The objective of this theoretical work was to derive solutions to static ferromagnetic problems that include current-carrying coils, uniform inducing fields, and linear and homogeneous ferromagnetic bodies. The solutions are intended to be used as classical benchmark validation problems for comparison with solutions to ferromagnetic problems obtained by various numerical techniques such as the finite difference method, the finite element method, and the integral equation iterative solution method.

APPROACH

After deriving the governing differential equations from Maxwell's equations for classical magnetostatic field theory, the method of separation of variables was employed to obtain the problem solution.

RESULTS

The magnetic induction was derived for several configurations of ferromagnetic spherical and prolate spheroidal bodies (hollow and solid) with internal and/or external infinitesimally thin spherical and spheroidal current bands, respectively. The magnetic induction is presented for ferromagnetic spherical and spheroidal bodies in a constant inducing field of arbitrary orientation. The ferromagnetic bodies were assumed to be linear and homogeneous. The reduction of the current band problem solutions to that of a current band in a vacuum is shown when the permeability of the ferromagnetic body is allowed to approach that of a vacuum. The application of the superposition principle, to obtain a total magnetic field solution for the case of a ferromagnetic body (hollow or solid) surrounding and/or surrounded by a current band and immersed in a uniform inducing field of arbitrary direction, is discussed.

ABSTRACT

Magnetic induction is calculated for several configurations of ferromagnetic spherical and prolate spheroidal bodies (hollow and solid) with internal and/or external infinitesimally thin spherical and spheroidal current bands, respectively. Magnetic induction is presented for ferromagnetic spherical and spheroidal bodies in a constant inducing field of arbitrary orientation. The ferromagnetic bodies are assumed to be linear and homogeneous. The reduction of the current band problem solutions to that of a current band in vacuum is shown when the permeability of the ferromagnetic body is allowed to approach that of vacuum. The application of the superposition principle to obtain a total magnetic field solution for the case of a ferromagnetic body (hollow or solid) surrounding and/or surrounded by a current band and immersed in a uniform inducing field of arbitrary direction, is discussed.

ADMINISTRATIVE INFORMATION

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INTRODUCTION

It is well known that exact analytical solutions of Maxwell's equations using classical formulation have been limited to body shapes and inhomogeneities that conform to a few separable coordinate systems. With the application of modern digital computers and numerical methods to obtain solutions of many magnetostatic field problems for practical applications, the need for classical benchmark validation problems arose. This theoretical report presents solutions of Maxwell's equations for magnetostatic problems. It summarizes twelve different problem solutions and discusses how to obtain the total field solution to many others through the application of the superposition principle. Many of these problem solutions may be used as benchmark type classical solutions and for research in studying magnetostatic effects. In addition, the solution techniques and verification methods presented in this report show the fundamental techniques of solving magnetostatic boundary value problem solutions of Laplace's and Poisson's equations for spherical and prolate spheroidal coordinate systems.

COORDINATE SYSTEMS

SPHERICAL COORDINATE SYSTEM

The spherical coordinate system is formed by the intersection of coordinate surfaces of concentric spheres, cones with apexes at the center of the spheres, and half planes emerging from the axis of the cone. The three coordinates of a point are the radius r of a sphere, the half-angle θ of the cone, and the angle ψ between a half-plane and the x axis. Figure 1 depicts the spherical coordinate system. With each point in the spherical coordinate system, there are associated three mutually perpendicular unit vectors $\hat{e}_r$, $\hat{e}_\theta$, and $\hat{e}_\psi$.

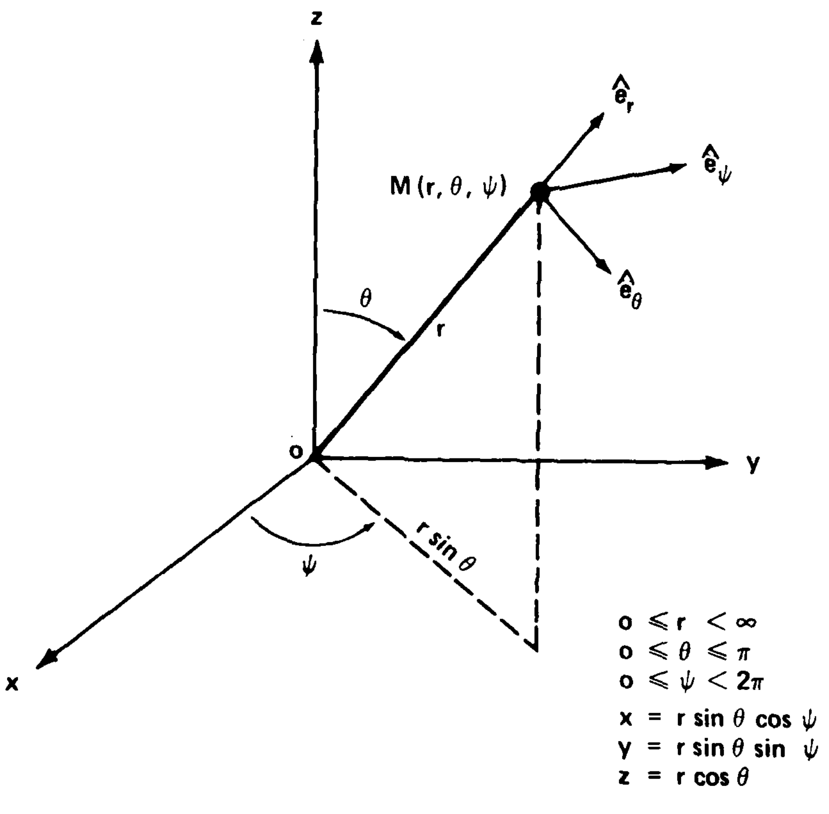

Figure 1 - Spherical Coordinate System
and the Corresponding Unit Vectors

PROLATE SPHEROIDAL COORDINATE SYSTEM

The prolate spheroidal coordinate system can be formed by rotating the two-dimensional elliptic coordinate system, whose traces in a plane are confocal ellipses and hyperbolas, about the major axis of the ellipse.[1,2]

Flammer[2] notes that it is customary to make the z-axis the axis of revolution. Figure 2 depicts the three-dimensional prolate spheroidal coordinate system. In this case, the coordinate surfaces are: prolate spheroids for η = constant; hyperboloids of two sheets for θ = constant; meridian planes for ψ = constant. The prolate spheroidal coordinates shown in Figure 2 are related to rectangular coordinates by the following transformation equations:

$$x = a \sinh \eta \sin \theta \cos \psi \tag{1a}$$

$$y = a \sinh \eta \sin \theta \sin \psi \tag{1b}$$

$$z = a \cosh \eta \cos \theta \tag{1c}$$

where $0 \leq \eta < \infty$

$0 \leq \theta \leq \pi$

$0 \leq \psi < 2\pi$

We have denoted the interfocal distance by 2a and the prolate spheroidal coordinates by (η,θ,ψ).

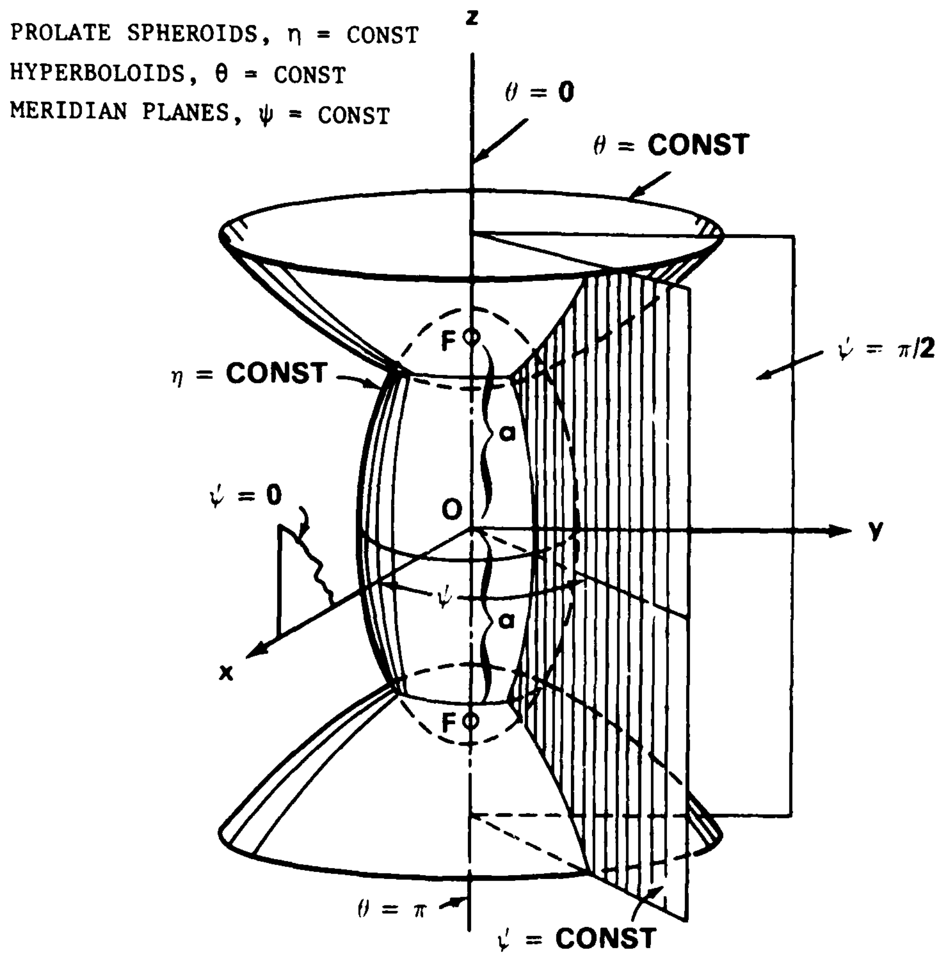

Figure 2 - Prolate Spheroidal Coordinate System

BASIC EQUATIONS

FIELD EQUATIONS

The formulation of the present boundary value problems implies the solution of Maxwell's equations for each medium subject to the classical boundary conditions. Starting with the general form of Maxwell's equations and the constitutive relations between $\overline{E}$ and $\overline{D}$ and between $\overline{B}$ and $\overline{H}$ as given below, a general solution may be derived.

$$\overline{\nabla} \times \overline{H} = \overline{J} + \frac{\partial \overline{D}}{\partial t} \qquad \overline{\nabla} \times \overline{E} = -\frac{\partial \overline{B}}{\partial t}$$

$$\overline{\nabla} \cdot \overline{D} = \rho \qquad \overline{\nabla} \cdot \overline{B} = 0 \tag{2a)*}$$

$$\overline{D} = \varepsilon_o \overline{E} + \overline{P} \qquad \overline{B} = \mu_o (\overline{H} + \overline{M}) \tag{2b}$$

where $\overline{H}$ = magnetic field intensity (A/m)**

$\overline{J}$ = electric current density (A/m^2)

$\overline{D}$ = electric flux density (C/m^2)

$\overline{E}$ = electric field intensity (V/m)

$\overline{B}$ = magnetic flux density (T or Wb/m^2)

ρ = free charge density (C/m^3)

$\overline{P}$ = polarization (C/m^2)

$\overline{M}$ = magnetization (A/m)

ε_o = permittivity of vacuum = 8.85 pF/m

μ_o = permeability of vacuum = 400π nH/m

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*The del operator $\overline{\nabla}$ is defined with respect to the rectangular coordinate system and is strictly valid in a rectangular coordinate system only. Very often $\overline{\nabla}$x and $\overline{\nabla}$ · are used generally as equivalent symbols for curl and divergence. This use is followed in this report.

**Definitions of symbols are given on page vii.

For the magnetostatic case, the applicable Maxwell's Equations (2a) reduce to

$$\overline{\nabla} \times \overline{H} = \overline{J} \qquad \overline{\nabla} \cdot \overline{B} = 0 \tag{3a}$$

and the constitutive relation from Equations (2b) is

$$\overline{B} = \mu_o(\overline{H}+\overline{M}) \tag{3b}$$

In general, for ferromagnetic materials, $\overline{B}$ is a nonlinear function of $\overline{H}$

$$\overline{B} = f(\overline{H}) \tag{4}$$

where, as shown in Figure 3a, $\overline{B}$ is not a single valued function of $\overline{H}$. The function $f(\overline{H})$ depends upon the magnetic history of the material, that is, how the metal became magnetized. This is referred to as hysteresis. It is also noted that any magnetic property of a ferromagnetic material has meaning only if it is considered together with its complete magnetic history.

In certain practical engineering problems, the variation in the magnetic field intensity is small, and the functional relationship between $\overline{B}$ and $\overline{H}$ is approximately linear (see Figure 3b). For the linear case where the material is isotropic, the magnetic induction $\overline{B}$ is related to the field intensity $\overline{H}$ by the relationship

$$\overline{B} = \mu_o\ (\chi_m+1)\ \overline{H} = \mu_o\mu_r\ \overline{H} = \mu\overline{H} \tag{5}$$

where χ_m = magnetic susceptibility (dimensionless)

μ = magnetic permeability of media (H/m)

(χ_m+1) = μ_r = relative permeability (dimensionless)

μ_o = permeability of vacuum = 400π nH/m

This report assumes that the ferromagnetic bodies have isotropic and linear material properties.

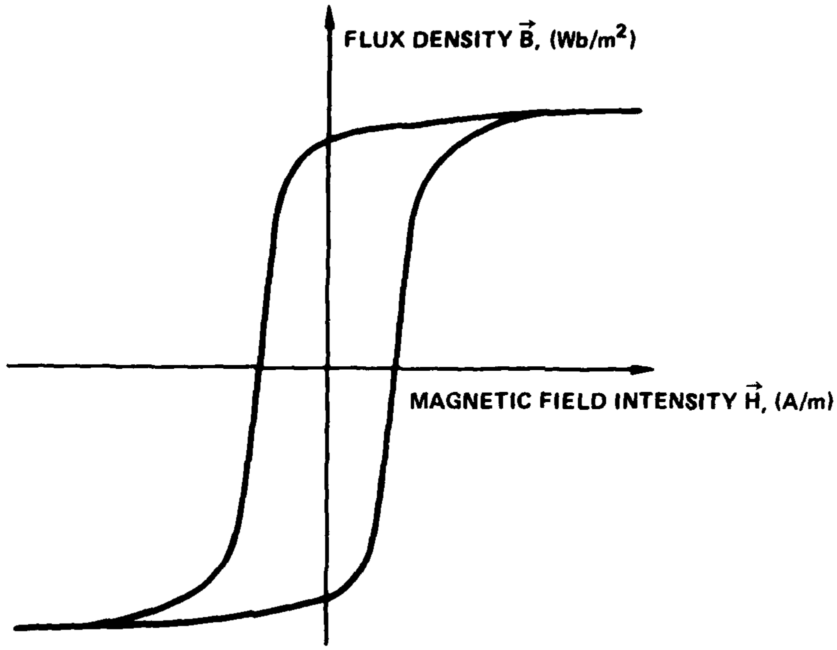

Figure 3a - Curve for a Ferromagnetic Material

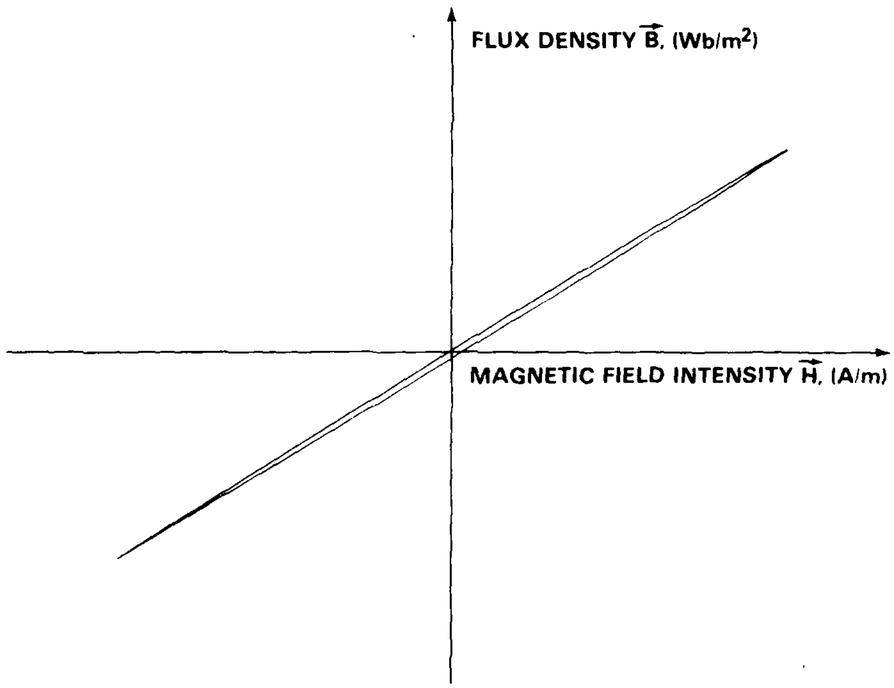

Figure 3b - Curve for a Ferromagnetic Material
at Low Inducing Fields

Figure 3 - Typical Magnetization Curve

SUPERPOSITION PRINCIPLE

Maxwell's Equations (2a) are linear partial differential equations. As a consequence of this linearity, the superposition principle states that, generally, any sum of the solutions of Maxwell's equations is again their solution. Combined with the uniqueness theorem, which states that only one solution of Maxwell's equation satisfies any set of prescribed boundary conditions, the superposition principle justifies any series or sum solution of Maxwell's equations.

Thus, if one desires to find the magnetic field solution to a system consisting of a ferromagnetic body in a uniform field and in the presence of current carrying conductors, the superposition principle may be applied. The magnetic field solution for a ferromagnetic body in a uniform field only is obtained first, then the magnetic field solution for the same ferromagnetic body in the presence of the current carrying conductors only is determined. The total magnetic field solution is then the sum of the two independent solutions. This technique allows, for example, one to find the total field solution for a hollow prolate spheroid immersed in a uniform field and surrounded by a current band.

MAGNETIC INDUCTION OF BODIES IN UNIFORM FIELDS

For the case of a ferromagnetic body of permeability μ in a uniform field in the absence of current carrying conductors, Maxwell's Equations (2a) reduce to

$$\overline{\nabla} \times \overline{H} = 0 \tag{6a}$$

$$\overline{\nabla} \cdot \overline{B} = 0 \tag{6b}$$

Because the curl of the gradient of any scalar function f is found to be identically zero $[\overline{\nabla}\times\overline{\nabla}f=0]$ the magnetic field intensity $\overline{H}$ is derivable as the gradient of a scalar potential ϕ_m. That is

$$\overline{H} = -\overline{\nabla}\phi_m \tag{7}$$

where ϕ_m is the magnetic scalar potential (in amperes). Using Equations (5) and (7) to find the magnetic flux density and substituting the result into Equation (6b) reduces to

$$\nabla^2\phi_m = 0 \tag{8}$$

which is known as Laplace's equation. This is the governing differential equation for the problem of a body immersed in a uniform field.

The general boundary conditions to be satisfied at the interface of dissimilar materials may be derived from the limiting integral form of Maxwell's equations and are given by

$$\overline{n}_{12} \cdot (\overline{B}_2-\overline{B}_1) = 0 \quad \text{or } B_{n1} = B_{n2} \quad \text{or} \quad \mu_1 \frac{\partial\phi_{m1}}{\partial n} = \mu_2 \frac{\partial\phi_{m2}}{\partial n} \tag{9a}$$

$$\overline{n}_{12} \times (\overline{H}_2-\overline{H}_1) = 0 \quad \text{or } H_{t2} = H_{t1} \quad \text{or} \quad \phi_{m1} = \phi_{m2} \tag{9b}$$

where the subscripts 1 and 2 indicate the media under consideration, and $\overline{n}_{12}$ denotes the unit vector normal to the interface and directed from medium 1 into medium 2.

MAGNETIC INDUCTION OF BODIES DUE TO CURRENT CARRYING CONDUCTORS

The divergenceless ($\overline{\nabla}\cdot\overline{B}=0$) nature of the magnetic flux density in conjunction with the fact that the divergence of the curl of any vector function is zero $[\overline{\nabla}\cdot(\overline{\nabla}\mathrm{x}\overline{F})=0]$ allows the introduction of the magnetic vector potential field ($\overline{A}$)

$$\overline{B} = \overline{\nabla}\mathrm{x}\overline{A} \tag{10}$$

where $\overline{A}$ is the magnetostatic vector potential function in webers per meter. The substitution of Equation (10) into Equation (3a) gives the fundamental equation of the vector potential of the magnetostatic field.

$$\frac{1}{\mu}\,\overline{\nabla} \times (\overline{\nabla} \times \overline{A}) - (\overline{\nabla} \times \overline{A}) \times \overline{\nabla}\,\frac{1}{\mu} = \overline{J} \tag{11}$$

For homogeneous materials, as assumed in this report, the magnetic permeability is spatially invariant. Hence

$$\overline{\nabla}\,\frac{1}{\mu} = 0 \tag{12}$$

and Equation (11) reduces to

$$\overline{\nabla} \times \overline{\nabla} \times \overline{A} = \mu \overline{J} \tag{13}$$

Using the vector identity

$$\overline{\nabla} \times \overline{\nabla} \times \overline{A} = \overline{\nabla}\,(\overline{\nabla} \cdot \overline{A}) - \nabla^2 \overline{A} \tag{14}$$

Equation (13) becomes

$$\overline{\nabla}(\overline{\nabla} \cdot \overline{A}) - \nabla^2 \overline{A} = \mu \overline{J} \tag{15}$$

The magnetostatic vector potential is characterized by the important property that its divergence can be conveniently chosen to be zero.

$$\overline{\nabla} \cdot \overline{A} = 0 \tag{16}$$

Equation (15) reduces to the vector Poisson differential equation.

$$\nabla^2 \bar{A} = - \mu \bar{J} \tag{17}$$

This is the governing equation for our calculations.

The general boundary conditions to be satisfied at the interfaces of stationary dissimilar media may be derived from the limiting integral forms of Maxwell's equations and are given by

$$\bar{n}_{12} \cdot (\bar{B}_2 - \bar{B}_1) = 0 \qquad \text{or} \qquad B_{n1} = B_{n2} \tag{18a}$$

$$\bar{n}_{12} \times (\bar{H}_2 - \bar{H}_1) = \bar{J}_s \qquad \text{or} \qquad H_{t2} - H_{t1} = J_s \tag{18b}$$

where the subscripts 1 and 2 indicate the media under consideration, and $\bar{n}_{12}$ denotes the unit vector normal to the interface and is directed from medium 1 into medium 2. In the case where the materials are linear and isotropic, Equations (18a) and (18b) become

$$\bar{n}_{12} \cdot (\mu_2 \bar{H}_2 - \mu_1 \bar{H}_1) = 0 \tag{18c}$$

$$\bar{n}_{12} \times \left(\frac{\bar{B}_2}{\mu_2} - \frac{\bar{B}_1}{\mu_1} \right) = \bar{J}_s \tag{18d}$$

where $\bar{J}_s$ is a true surface current density that may exist at the interface. At an interface where $\bar{J}_s = 0$, Equations (18b) and (18d) need to be modified accordingly.

SOLUTIONS FOR SPHERICAL BODIES

SOLID SPHERE OR SPHERICAL SHELL IN A UNIFORM FIELD OF ARBITRARY DIRECTION

Several important types of problems relating to magnetized bodies in an external magnetic field have been solved[3,4,5] by determining the solution to Laplace's equation for the magnetic scalar potential. Generally, these solutions have been derived for the case of the uniform external magnetic field in the direction of the z axis of a spherical coordinate system. Both constant external field problems, solid and shell, were solved and programmed on the digital computer by D.A. Nixon of the Center, for the case of an arbitrarily orientated external magnetic field.[6] The solutions found in Reference 6 were presented in Cartesian coordinates. The problem of finding the magnetic induction for an infinitesimally thin current band surrounding a spherical shell[7] can be generalized to include an external magnetic field. Linear superposition may be applied to find the solution in this case. Therefore, in Appendix A, the Cartesian expressions were converted to spherical coordinates to be compatible with other problem solutions in this section of the report.

SOLID SPHERE SURROUNDED BY AN INFINITESIMALLY THIN SPHERICAL CURRENT BAND

We now solve the boundary problem of a ferromagnetic sphere of radius R_1 and homogeneous permeability μ_2 surrounded by an infinitesimally thin current band of radius R_2 having a constant current density $\overline{J}$. Figure 4 identifies the three regions of interest. Regions II and III have a permeability equal to the permeability of vacuum μ_o, which for convenience will be labeled μ_1. The problem's spherical symmetry suggests that a spherical coordinate system such as that shown in Figure 1 be used in the problem solution.

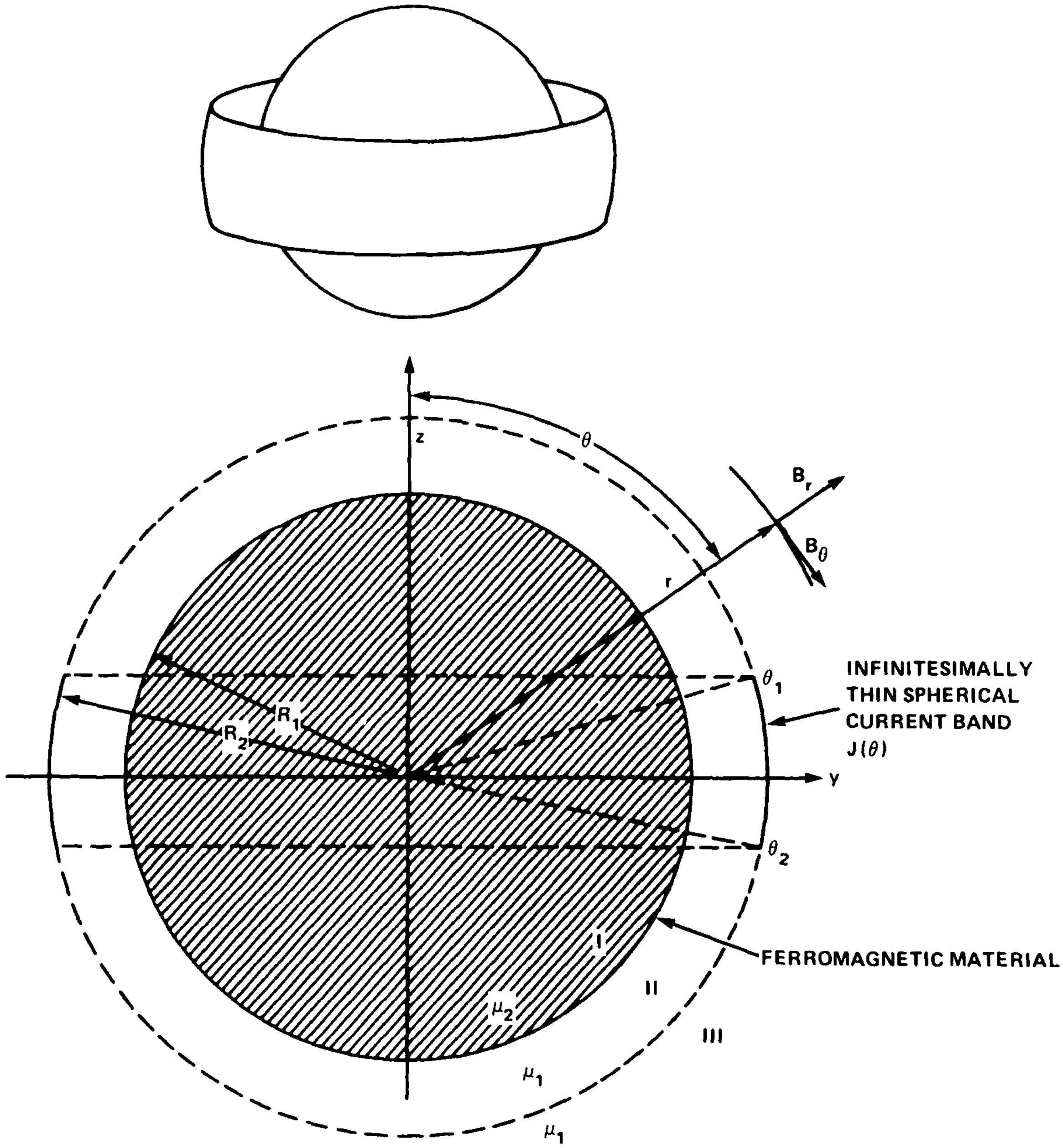

Figure 4 - Ferromagnetic Sphere Surrounded by an Infinitesimally Thin Current Band

Ampere's law states

$$\overline{\nabla} \times \overline{H} = \overline{J} \tag{19}$$

and, because $\overline{\nabla} \cdot \overline{B} = 0$, the induction $\overline{B}$ must be the curl of some vector field $\overline{A}$. The governing differential equation for $\overline{A}$, when homogeneous and linear materials are considered, is from Equation (11).

$$\text{✡}\,\overline{A} = -\mu\overline{J}^{\,*} \tag{20}$$

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*We note that a distinction is drawn between the operator ∇^2 called the scalar Laplacian operator and the vector Laplacian operator designated by ✡ . The vector Poisson's equation in rectangular coordinates can be treated as three uncoupled scalar equations as shown below.

$$\text{✡}\,\overline{A} = \hat{e}_x \left[\frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_x}{\partial y^2} + \frac{\partial^2 A_x}{\partial z^2}\right] + \hat{e}_y \left[\frac{\partial^2 A_y}{\partial x^2} + \frac{\partial^2 A_y}{\partial y^2} + \frac{\partial^2 A_y}{\partial z^2}\right]$$

$$+ \hat{e}_z \left[\frac{\partial^2 A_z}{\partial x^2} + \frac{\partial^2 A_z}{\partial y^2} + \frac{\partial^2 A_z}{\partial z^2}\right] = \hat{e}_x J_x + \hat{e}_y J_y + \hat{e}_z J_z$$

where $\text{✡}A_i = \nabla^2 A_i = \mu J_i$ for $i = x,y,z$. However, if the vector Poisson's equation is resolved into orthogonal components in other coordinate systems, the differential operation mixes the components together giving coupled equations as shown below for spherical coordinates.

$$\nabla^2 \overline{A} = \hat{e}_r \left[\frac{\partial^2 A_r}{\partial r^2} + \frac{2}{r}\frac{\partial A_r}{\partial r} - \frac{2}{r^2} A_r + \frac{1}{r^2}\frac{\partial^2 A_r}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial A_r}{\partial \theta} \right.$$

$$\left. + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 A_r}{\partial \psi^2} - \frac{2}{r^2}\frac{\partial A_\theta}{\partial \theta} - \frac{2\cot\theta}{r^2} A_\theta - \frac{2}{r^2 \sin\theta}\frac{\partial A_\psi}{\partial \psi} \right]$$

$$+ \hat{e}_\theta \left[\frac{\partial^2 A_\theta}{\partial r^2} + \frac{2}{r}\frac{\partial A_\theta}{\partial r} - \frac{A_\theta}{r^2 \sin^2\theta} + \frac{1}{r^2}\frac{\partial^2 A_\theta}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial A_\theta}{\partial \theta} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 A_\theta}{\partial \psi^2} \right.$$

$$\left. + \frac{2}{r^2}\frac{\partial A_r}{\partial \theta} - \frac{2\cot\theta}{r^2 \sin\theta}\frac{\partial A_\psi}{\partial \psi} \right] + \hat{e}_\psi \left[\frac{\partial^2 A_\psi}{\partial r^2} + \frac{2}{r}\frac{\partial A_\psi}{\partial r} - \frac{1}{r^2 \sin^2\theta} A_\psi + \frac{1}{r^2}\frac{\partial^2 A_\psi}{\partial \theta^2} \right.$$

$$\left. + \frac{\cot\theta}{r^2}\frac{\partial A_\psi}{\partial \theta} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 A_\psi}{\partial \psi} + \frac{2}{r^2 \sin\theta}\frac{\partial A_r}{\partial \psi} + \frac{2\cot\theta}{r^2 \sin\theta}\frac{\partial A_\theta}{\partial \psi} \right]$$

$$= J_r \hat{e}_r + J_\theta \hat{e}_\theta + J_\psi \hat{e}_\psi$$

The general expression in spherical coordinates for a current density is

$$\overline{J} = \hat{e}_r J_r + \hat{e}_\theta J_\theta + \hat{e}_\psi J_\psi \tag{21}$$

where the $\hat{e}$ defines unit orthogonal vectors. For stationary currents in vacuum, the vector potential function that satisfies Equation (20) is given by

$$\overline{A} = \frac{\mu_o}{4\pi} \int_V \frac{\overline{J}}{r'} \, dv \qquad (22)$$

where dv = elemental volume in the current-carrying region

r' = distance between the field point where $\overline{A}$ is being determined and dv at the source point.

From Equation (22) we see that the elemental vector potential $d\overline{A}$, due to a current element $\overline{J}dv$, is in the same direction as $\overline{J}$. It is well known from this fact that the lines of the magnetic vector potential $\overline{A}$ are circles centered about the coil or loop axis. The magnitude of $\overline{A}$ along such a circle is constant, which means that $\overline{A}$ is a function of the spherical coordinates r and θ only. Therefore, we know in advance for this problem that A_ψ is the only component of $\overline{A}$ existing at the field point. The infinitesimally thin band of current, shown in Figure 4, has only an azimuthal or ψ component, which is a function of r and θ, and lies on the boundary between regions II and III (i.e., $r=R_2$). For this current, Equation (21) reduces to

$$\overline{J} = \begin{cases} 0 & , \text{ if } \theta < \theta_1 \text{ or } \theta > \theta_2 \\ \hat{e}_\psi J_\psi(\theta) & , \text{ if } \theta_1 \leq \theta \leq \theta_2 \end{cases} \qquad (23)$$

Therefore, Equation (20) has only an azimuthal component and can be expressed as

$$\text{✡} A_\psi = \text{✡} A_\psi(r,\theta) = 0 \text{ (in regions I through III)} \qquad (24)$$

When the vector Laplacian ∇^2 is expanded in spherical coordinates, Equation (24) can be written as

$$\frac{\partial^2 A_\psi}{\partial r^2} + \frac{2}{r}\frac{\partial A_\psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 A_\psi}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial A_\psi}{\partial \theta} - \frac{A_\psi}{r^2 \sin^2\theta} = 0 \quad \text{in regions I through III} \tag{25}$$

In order to solve Equation (25) it is necessary to obtain the general solution in regions I through III. Thus, by multiplying Equation (25) by r^2 we obtain

$$\frac{r^2 \partial^2 A_\psi}{\partial r^2} + \frac{2r\partial A_\psi}{\partial r} + \frac{\partial^2 A_\psi}{\partial \theta^2} + \cot\theta\,\frac{\partial A_\psi}{\partial \theta} - \frac{A_\psi}{\sin^2\theta} = 0 \tag{26}$$

Applying the method of separation of variables, let us assume that A_ψ can be expressed as a product of two functions

$$A_\psi = R(r)\Theta(\theta) \tag{27}$$

where $R(r)$ is a function of r only and $\Theta(\theta)$ of θ only. Substituting this form of the vector potential A_ψ into Equation (26), we have, after separation of variables

$$\frac{d^2R(r)}{dr^2} + \frac{2}{r}\frac{dR(r)}{dr} - \frac{p(p+1)R(r)}{r^2} = 0 \tag{28a}$$

$$\frac{d^2\Theta(\theta)}{d\theta^2} + \cot\theta\,\frac{d\Theta(\theta)}{d\theta} + \left[p(p+1) - \frac{1}{\sin^2\theta}\right]\Theta(\theta) = 0 \tag{28b}$$

where the separation constant is p(p+1) and p is an integer from one to infinity. The differential equation

$$\frac{d^2\Theta}{d\theta^2} + \cot\theta\,\frac{d\Theta}{d\theta} + \left[p\,(p+1) - \frac{m^2}{\sin^2\theta}\right]\Theta = \theta \tag{29}$$

has, as a general solution,

$$\Theta(\theta) = \Theta_p(\theta) = C_p P_p^m\,(\cos\theta) + D_p Q_p^m\,(\cos\theta) \tag{30}$$

Comparison of Equations (28b) and (29) shows that in Equation (29) $m^2 = 1$. This requires that m always be unity. The solutions of Equations (28a) and (28b) are then expressed as

$$R(r) = R_p(r) = A_p' r^p + B_p' r^{-(p+1)} \tag{31a}$$

$$\Theta(\theta) = \Theta_p(\theta) = C_p P_p^1(\cos\theta) + D_p Q_p^1(\cos\theta) \tag{31b}$$

The associated Legendre functions of the first and second kind are designated as $P_p^m(\cos\theta)$ and $Q_p^m\,(\cos\theta)$, respectively. Therefore, the general solution of

Equation (25) in regions I through III may be formed from the product of the solutions in Equation (31), which yields

$$A_\psi = R(r)\Theta(\theta) = \sum_{p=1}^{\infty} R_p(r)\Theta_p(\theta) \tag{32a}$$

$$= \sum_{p=1}^{\infty} \left(A'_p r^p + \frac{B'_p}{r^{(p+1)}}\right) \left(C_p P_p^1(\cos\theta) + D_p Q_p^1(\cos\theta)\right) \tag{32b}$$

In the spherical case, associated Legendre functions of the second kind are infinite at $\cos\theta = \pm 1$, and, thus, cannot be included when the region under consideration includes the symmetry axis. Therefore, the constant D_p must be set equal to zero, and Equation (32) reduces to

$$A_\psi = \sum_{p=1}^{\infty} \left(A_p r^p + \frac{B_p}{r^{(p+1)}}\right) P_p^1(\cos\theta) \tag{33}$$

where $A_p = A'_p C_p$ and $B_p = B'_p C_p$.

The form of the potential in each of the regions (I through III) is determined from Equation (33). These magnetostatic vector potentials in regions I through III are:

$$A_I = A_{\psi I} = \sum_{p=1}^{\infty} \left(A_{p1} r^p\right) P_p^1(\cos\theta) \tag{34a}$$

$$A_{II} = A_{\psi II} = \sum_{p=1}^{\infty} \left[A_{p2} r^{p} + \frac{B_{p2}}{r^{(p+1)}} \right] P_p^1(\cos\theta) \tag{34b}$$

$$A_{III} = A_{\psi III} = \sum_{p=1}^{\infty} \left[\frac{B_{p3}}{r^{(p+1)}} \right] P_p^1(\cos\theta) \tag{34c}$$

where, for the $A_{\psi I}$ component, $B_{p1} = 0$ because at $r = 0$ the potential must be finite and, for the $A_{\psi III}$ component, $A_{p3} = 0$ because as r approaches infinity the potential must remain finite.

At each interface, the basic laws of magnetostatics in Equations (3a) reduce to boundary conditions on $\overline{B}$ and $\overline{H}$ that can be used to evaluate the four constants in Equation (34). From Equations (18a), the normal component of $\overline{B}$ across each boundary must be continuous, i.e., $(\overline{B}_2 - \overline{B}_1) \cdot \overline{n}_{12} = 0$ where the quantity $\overline{n}_{12}$ is the unit outward normal to the surface. This provides the following boundary conditions which must be satisfied by the solution in Equation (34) for each region.

$$B_{rI} = B_{rII} \text{ at } r = R_1 \tag{35a}$$

$$B_{rII} = B_{rIII} \text{ at } r = R_2 \tag{35b}$$

The normal component of the magnetic field B_r is expressed in terms of the vector potential as

$$B_r = (\overline{\nabla} x \overline{A})_r \tag{36a}$$

$$B_r = \frac{1}{r \sin\theta} \frac{\partial}{\partial\theta} (\sin\theta \, A_\psi) \tag{36b}$$

$$\text{where} \quad \overline{B} = \overline{\nabla} x \overline{A} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{e}_r & \hat{e}_\theta r & \hat{e}_\psi r \sin\theta \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\psi} \\ 0 & 0 & A_\psi r \sin\theta \end{vmatrix}$$

However, because the vector potentials in each region are functions of P_p^1 (cos θ) we can simplify Equation (35) to constraints on A_ψ

$$A_I = A_{II} \text{ at } r = R_1 \tag{37a}$$

$$A_{II} = A_{III} \text{ at } r = R_2 \tag{37b}$$

The second set of boundary conditions is obtained from Equation (18b). The tangential component of $\overline{H}$ across each boundary must satisfy the relationship

$$\overline{n}_{12} \times (\overline{H}_2 - \overline{H}_1) = \overline{J}_s \tag{38}$$

where $\overline{J}_s$ (which equals $\overline{J}(\theta)$) is the real surface current density in the limit of vanishing width between the two regions. Using the relationship $\overline{B} = \mu\overline{H}$, Equation (38) may be expressed as

$$\frac{B_{\theta 2}}{\mu_2} - \frac{B_{\theta 1}}{\mu_1} = J(\theta) \tag{39}$$

Referring to the curl in Equation (36), we can write B_θ as

$$B_\theta = (\overline{\nabla} x \overline{A})_\theta = -\frac{1}{r}\frac{\partial}{\partial r}\left[rA_\psi\right] \tag{40}$$

From Equations (38), (39), and (40), the tangential components in regions I through III must satisfy the relationships

$$-\frac{1}{\mu_1}\ \frac{1}{r}\ \frac{\partial}{\partial r}(rA_{II}) + \frac{1}{\mu_2}\ \frac{1}{r}\ \frac{\partial}{\partial r}(rA_I) = 0 \quad \text{at } r = R_1 \tag{41a}$$

$$-\frac{1}{\mu_1}\ \frac{1}{r}\ \frac{\partial}{\partial r}(rA_{III}) + \frac{1}{\mu_1}\ \frac{1}{r}\ \frac{\partial}{\partial r}(rA_{II}) = J(\theta) \quad \text{at } r = R_2 \tag{41b}$$

The general expressions for the potentials in each region (Equation (34)) are then substituted into the boundary conditions (Equations (37) and (41)) and solved for the constants A_{pi} and B_{pi}. There are four algebraic equations with four unknowns and the potential in each region can then be specifically determined. The four boundary value equations that must be solved for the coefficients are given below (where the index p is odd only and understood to take on values from 1 to ∞). It is noted that the current $J_\psi(\theta)$ must be expanded into a set of associated Legendre functions in order to evaluate the constants A_{pi} and B_{pi}. The detailed expansion is in the section of Reference 7 entitled "Expansion of the Current $(J_\psi(\theta))$ in Associated Lengendre Polynomials."

$$A_{p1}R_1{}^p = \left[A_{p2}R_1^p + B_{p2}R_1{}^{-(p+1)}\right] \tag{42a}$$

$$\left[A_{p2}R_2^p + B_{p2}R_2{}^{-(p+1)}\right] = B_{p3}\left[R_2{}^{-(p+1)}\right] \tag{42b}$$

$$-\frac{1}{\mu_1}\left[A_{p2}(p+1)R_1{}^{(p-1)} - B_{p2}pR_1{}^{-(p+2)}\right] + \frac{1}{\mu_2}\left[A_{p1}(p+1)R_1{}^{(p-1)}\right] = 0 \tag{42c}$$

$$\frac{1}{\mu_1}\left[B_{p3}pR_2{}^{-(p+2)}\right] + \frac{1}{\mu_1}\left[A_{p2}(p+1)R_2{}^{(p-1)} - pB_{p2}R_2{}^{-(p+2)}\right] = \frac{J_p(\theta)}{P_p^1(\cos\theta)} \tag{42d}$$

The solution of these equations to obtain the coefficients yields (a detailed derivation is given in Appendix B):

$$B_{p2} = \frac{\left(1 - \frac{\mu_2}{\mu_1}\right) \frac{\mu_1 J_p(\theta)}{P_p^1(\cos\theta)}}{\left(-R_1^{-(2p+1)}\right)(2p+1) R_2^{(p-1)} \left(1 + \frac{\mu_2}{\mu_1}\left(\frac{p}{p+1}\right)\right)} \tag{43a}$$

$$A_{p2} = \frac{\mu_1 J_p(\theta)}{(2p+1) R_2^{p-1} P_p^1(\cos\theta)} \tag{43b}$$

$$A_{p1} = \frac{\left[\mu_1 J_p(\theta)\right]}{(2p+1) R_2^{(p-1)} P_p^1(\cos\theta)} - \frac{\left[\left(1 - \frac{\mu_2}{\mu_1}\right) \frac{\mu_1 J_p(\theta)}{P_p^1(\cos\theta)}\right]}{(2p+1) R_2^{(p-1)} \left(1 + \frac{\mu_2}{\mu_1}\left(\frac{p}{p+1}\right)\right)} \tag{43c}$$

$$B_{p3} = \frac{\frac{\mu_1 J_p(\theta)}{P_p^1(\cos\theta)} R_2^{(2p+1)}}{(2p+1) R_2^{(p-1)}} - \frac{\left(1 - \frac{\mu_2}{\mu_1}\right) \frac{\mu_1 J_p(\theta)}{P_p^1(\cos\theta)}}{R_1^{-(2p+1)} (2p+1) R_2^{(p-1)} \left(1 + \frac{\mu_2}{\mu_1}\left(\frac{p}{p+1}\right)\right)} \tag{43d}$$

The coefficients A_{pi} and B_{pi} can be determined from Equations (43), and Equations (34) can now be used to specify the potentials A_I, A_{II}, and A_{III} in regions I through III. The normal (B_r) and tangential (B_θ) components of the magnetic induction in regions I through III can be determined, by using Equations (36) and (40), to be:

$$B_{\theta I} = -\sum_{p=1}^{\infty} (p+1)\, A_{p1} r^{(p-1)}\, P_p^1 (\cos\theta) \tag{44a}$$

$$B_{rI} = \sum_{p=1}^{\infty} \frac{1}{\sin\theta} A_{p1} r^{(p-1)} \frac{\partial}{\partial\theta} \left[\sin\theta\, P_p^1 (\cos\theta)\right] \tag{44b}$$

$$B_{\theta II} = -\sum_{p=1}^{\infty} \left[A_{p2}(p+1) r^{(p-1)} - pB_{p2} r^{-(p+2)}\right] P_p^1 (\cos\theta) \tag{44c}$$

$$B_{rII} = \sum_{p=1}^{\infty} \frac{1}{\sin\theta} \left[A_{p2} r^{(p-1)} + B_{p2} r^{-(p+2)}\right] \frac{\partial}{\partial\theta} \left(\sin\theta\, P_p^1 (\cos\theta)\right) \tag{44d}$$

$$B_{\theta III} = \sum_{p=1}^{\infty} \left[pB_{p3} r^{-(p+2)}\right] P_p^1 (\cos\theta) \tag{44e}$$

$$B_{rIII} = \sum_{p=1}^{\infty} \frac{1}{\sin\theta} B_{p3} r^{-(p+2)} \frac{\partial}{\partial\theta} \left(\sin\theta\, P_p^1 (\cos\theta)\right) \tag{44f}$$

The magnetic vector potential, $A_{\psi I}$ in the inner region and $A_{\psi II}$ in the outer region, are derived for an infinitesimally thin current band in a homogeneous medium of permeability μ_1, in Reference 7. The coefficients A_{pi} $(i = 1,2)$ and B_{pi} $(i = 2,3)$, for the vector potentials for the present ferromagnetic sphere problem, reduce to the coefficients of the potentials in the two regions of the simple current band problem when the permeability of the sphere μ_2 approaches that of the surrounding medium μ_1. This shows that the solutions of the above ferromagnetic current problem have the correct mathematical form.

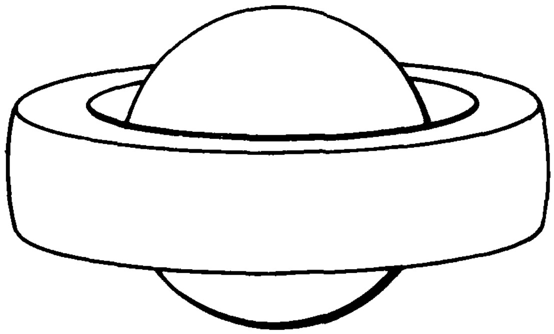
We note that derivation of the solution for the problem of a ferromagnetic sphere surrounded by a coil of finite width is found in Reference 7. See Figure 5 for the geometry of the problem. The magnetic induction for this case is:

$$B_{\theta I} = -\sum_{p=1}^{\infty} (p+1)\, A_{p1} r^{(p-1)}\, P_p^1 (\cos\theta) \tag{45a}$$

$$B_{rI} = \frac{1}{\sin\theta} \sum_{p=1}^{\infty} A_{p1} r^{(p-1)} \frac{\partial}{\partial\theta}\left[\sin\theta\, P_p^1 (\cos\theta)\right] \tag{45b}$$

$$B_{\theta II} = \sum_{p=1}^{\infty} \left[-(p+1)\, A_{p2} r^{(p-1)} + pB_{p2} r^{-(p+2)}\right] P_p^1 (\cos\theta) \tag{45c}$$

$$B_{rII} = \frac{1}{\sin\theta} \sum_{p=1}^{\infty} \left[A_{p2} r^{(p-1)} + \frac{B_{p2}}{r^{(p+2)}}\right] \frac{\partial}{\partial\theta}\left[\sin\theta\, P_p^1 (\cos\theta)\right] \tag{45d}$$



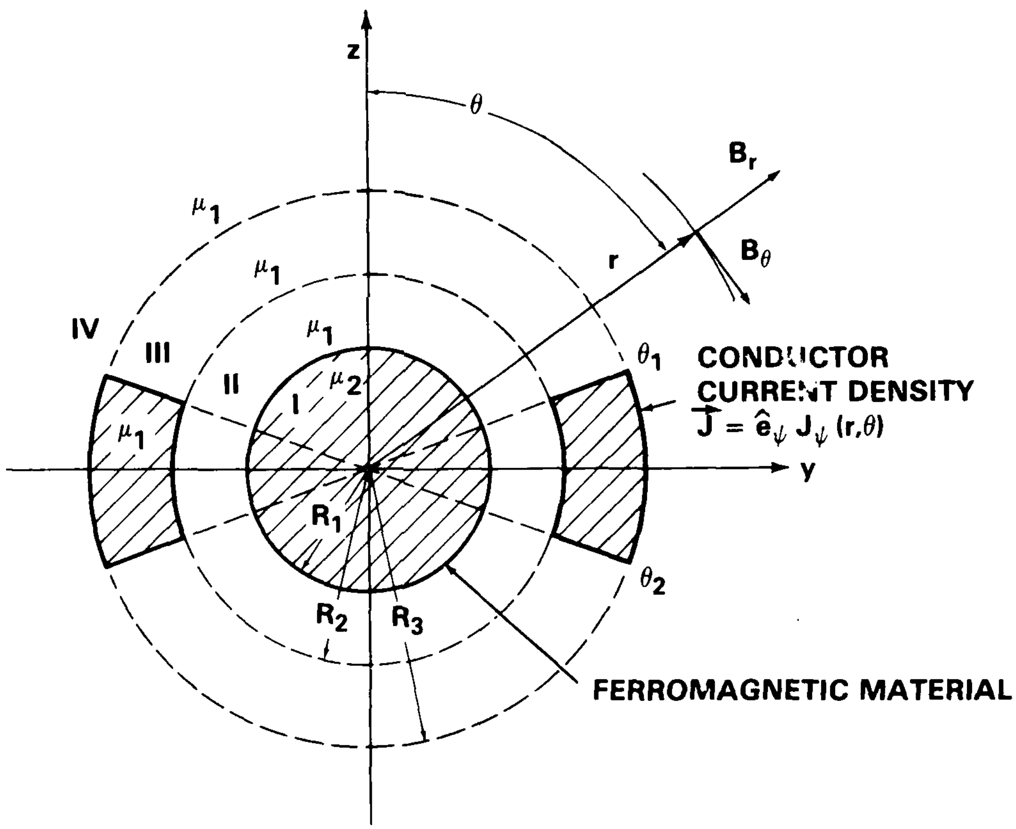


Figure 5 - Ferromagnetic Sphere Surrounded by a Coil of Finite Width (yz plane)

$$B_{\theta III} = \sum_{p=1}^{\infty} \left[-(p+1)\ A_{p3} r^{(p-1)} + pB_{p3} r^{-(p+2)}\right] P_p^1\ (\cos\theta)$$

$$- 3 \sum_{\substack{p\neq 2\\ p=1}}^{\infty} \left(\frac{\mu_1 JrK_p}{(p-2)(p+3)}\right) P_p^1\ (\cos\theta) \tag{45e}$$

$$B_{rIII} = \frac{1}{\sin\theta} \sum_{p=1}^{\infty} \left[A_{p3} r^{(p-1)} + \frac{B_{p3}}{r^{(p+2)}}\right] \frac{\partial}{\partial\theta} \left[\sin\theta\ P_p^1\ (\cos\theta)\right]$$

$$+ \frac{1}{\sin\theta} \sum_{\substack{p\neq 2\\ p=1}}^{\infty} \left[\frac{\mu_1 JrK_p}{(p-2)(p+3)}\right] \frac{\partial}{\partial\theta} \left[\sin\theta\ P_p^1\ (\cos\theta)\right] \tag{45f}$$

$$B_{\theta IV} = \sum_{p=1}^{\infty} \left[pB_{p4} r^{-(p+2)}\right] P_p^1\ (\cos\theta) \tag{45g}$$

$$B_{rIV} = \frac{1}{\sin\theta} \sum_{p=1}^{\infty} \left[B_{p4} r^{-(p+2)}\right] \frac{\partial}{\partial\theta} \left[\sin\theta\ P_p^1\ (\cos\theta)\right] \tag{45h}$$

The mathematical solution for B_{p3} in terms of known quantities, obtained in Appendix E of Reference 7, is given by

$$B_{p3} = \left\{ -K_p'(p+1)R_2^{(p-1)} - \frac{3\mu_1 JR_2K_p}{(p-2)(p+3)} + [X]\ (p+1)R_2^{(p-1)}K_p'' - (p)R_2^{-(p+2)}K_p'' \right\} \Big/$$

$$\left\{ (-p)R_2^{-(p+2)} - [Z][X](p+1)R_2^{(p-1)} + [Z](p)R_2^{-(p+2)} \right\} \tag{46a}$$

$$[X] = \frac{-R_1^{-(2p+1)}\left[1+\left(\frac{p}{p+1}\right)\frac{\mu_2}{\mu_1}\right]}{\left(1-\frac{\mu_2}{\mu_1}\right)} \tag{46b}$$

$$K_p' = -\frac{\mu_1 JK_pR_3^{-p+2}}{(p-2)(2p+1)} \tag{46c}$$

$$[Z] = \frac{R_2^{-(p+1)}}{\left([X]\ R_2^{p}+R_2^{-(p+1)}\right)} \tag{46d}$$

$$K_p'' = \frac{\frac{\mu_1 JR_2^{2}K_p}{(p-2)(p+3)} + K_p'R_2^{p}}{\left([X]R_2^{p}+R_2^{-(p+1)}\right)} \tag{46e}$$

The numerical values for the other coefficients can be obtained from the equations

$$A_{p3} = K_p' \tag{47a}$$

$$B_{p2} = B_{p3} \, [Z] + K_p'' \tag{47b}$$

$$A_{p2} = [X] \, B_{p2} \tag{47c}$$

$$A_{p1} = A_{p2} + B_{p2} R_1^{-(2p+1)} \tag{47d}$$

$$B_{p4} = A_{p3} R_3^{(2p+1)} + B_{p3} + \frac{\mu_1 J R_3^{(p+3)} K_p}{(p-2)(p+3)} \tag{47e}$$

SPHERICAL SHELL SURROUNDING AN INFINITESIMALLY THIN SPHERICAL CURRENT BAND

We now proceed to solve the boundary value problem of a ferromagnetic spherical shell of outer radius R_3, inner radius R_2, of homogeneous permeability μ_2, surrounding an infinitesimally thin current band of radius R_1 and having a current density $\overline{J}$. A constant density $\overline{J}$ is assumed. Figure 6 shows the four regions of interest. Regions I, II, and IV have a permeability equal to the permeability of vacuum μ_0 which, for convenience, will be labeled μ_1. The problem's spherical symmetry suggest that a spherical coordinate system, such as that shown in Figure 1, be used in the solution.

The details of treating problems of this type with spherical symmetry are discussed in the previous section of this report and in Reference 7.

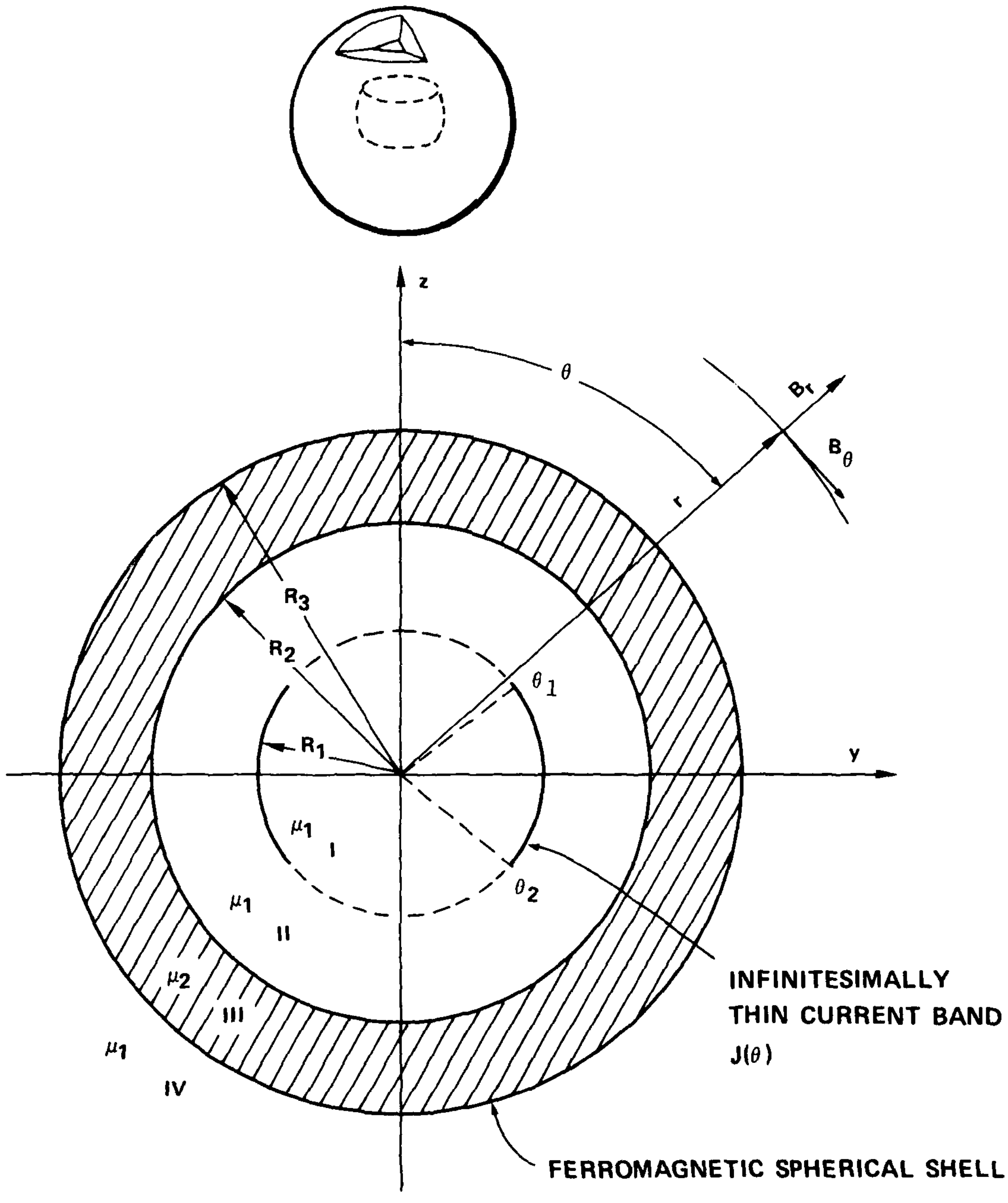


Figure 6 - Infinitesimally Thin Current Band Surrounded by a Ferromagnetic Spherical Shell

The partial differential equation that governs this problem is the azimuthal component of the vector Laplace's equation.

$$\Delta\!\!\!\!\nabla\, A_{\psi} = \Delta\!\!\!\!\nabla\, A_{\psi}(r,\theta) = 0 \quad \text{in regions I through IV} \tag{48}$$

When the vector Laplacian is expanded in spherical coordinates, Equation (48) can be written as

$$\frac{\partial^2 A_{\psi}}{\partial r^2} + \frac{2}{r}\frac{\partial A_{\psi}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 A_{\psi}}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial A_{\psi}}{\partial \theta} - \frac{A_{\psi}}{r^2 \sin^2\theta} = 0 \quad \text{(in regions I through IV)} \tag{49}$$

The general solution of the equation has the form

$$A_{\psi} = \sum_{p=1}^{\infty} \left(A_p r^p + \frac{B_p}{r^{(p+1)}} \right) P_p^1 (\cos\theta) \tag{50}$$

where A_p and B_p are constants and P_p^1 (cos θ) is the associated Legendre function of the first kind. The magnetostatic components of the vector potential in regions I through IV are

$$A_{\psi I} = \sum_{p=1}^{\infty} \left(A_{p1} r^p \right) P_p^1 (\cos\theta) \tag{51a}$$

$$A_{\psi II} = \sum_{p=1}^{\infty} \left(A_{p2} r^p + \frac{B_{p2}}{r^{(p+1)}} \right) P_p^1 (\cos\theta) \tag{51b}$$

$$A_{\psi III} = \sum_{p=1}^{\infty} \left(A_{p3} r^p + \frac{B_{p3}}{r^{(p+1)}} \right) P_p^1 (\cos \theta) \tag{51c}$$

$$A_{\psi IV} = \sum_{p=1}^{\infty} \left(\frac{B_{p4}}{r^{(p+1)}} \right) P_p^1 (\cos \theta) \tag{51d}$$

where, for the $A_{\psi I}$ component, $B_{p1} = 0$, because at $r = 0$ the potential must be finite, and, for the $A_{\psi IV}$ component, $A_{p4} = 0$, because as r approaches infinity the potential must remain finite.

At each interface, the basic laws of magnetostatics reduce to boundary conditions on $\overline{B}$ and $\overline{H}$ that can be used to evaluate the six constants in Equations (51). The first boundary condition states that the normal component of $\overline{B}$ across each boundary must be equal to $(\overline{B}_2-\overline{B}_1) \cdot \overline{n}_{12} = 0$. The vector quantity $\overline{n}_{12}$ is the unit outward normal to the surface (in the spherical case $\hat{e}_r$). Thus, the following boundary conditions must be satisfied by the four regions of the ferromagnetic spherical shell problem.

$$B_{rI} = B_{rII} \qquad \text{at } r = R_1 \tag{52a}$$

$$B_{rII} = B_{rIII} \qquad \text{at } r = R_2 \tag{52b}$$

$$B_{rIII} = B_{rIV} \qquad \text{at } r = R_3 \tag{52c}$$

The second boundary condition states that the tangential component of $\overline{H}$ across each boundary must satisfy the relationship

$$\overline{n}_{12} \times (\overline{H}_2-\overline{H}_1) = \overline{J}_s \tag{53}$$

where $\overline{J}_s$ (which equals $\overline{J}(\theta)$ in our case) is the true surface current density in the limit of the vanishing width between the two regions. Using the linear relationship $\overline{B} = \mu\overline{H}$, Equation (53) can be expressed in spherical coordinates as:

$$\frac{B_{\theta 2}}{\mu_2} - \frac{B_{\theta 1}}{\mu_1} = J(\theta) \tag{54}$$

The general expressions for the components of the vector potentials in each region A_ψ (Equations (51)) are then substituted into the boundary conditions (Equations (52) and (54)) and solved for the constants A_{pi} and B_{pi}. There are six algebraic equations with six unknowns, thus enabling the potential in each region to be specifically determined. The six boundary value equations that must be solved for the coefficients are given below (where the index p is odd only and understood to take on values from 1 to ∞).

As in the previous section, the component of the current $J_\psi(\theta)$ must be expanded into a set of associated Legendre functions in order to evaluate the constants A_{pi} and B_{pi}. (The detailed expansion of the azimuthal component of the current density is given in Reference 7).

$$A_{p1}R_1^p = \left[A_{p2}R_1^p + B_{p2}R_1^{-(p+1)}\right] \tag{55a}$$

$$\left[A_{p2}R_2^p + B_{p2}R_2^{-(p+1)}\right] = \left[A_{p3}R_2^p + B_{p3}R_2^{-(p+1)}\right] \tag{55b}$$

$$\left[A_{p3}R_3^p + B_{p3}R_3^{-(p+1)}\right] = B_{p4}\left[R_3^{-(p+1)}\right] \tag{55c}$$

$$-\frac{1}{\mu_1}\left[A_{p2}(p+1)R_1^{(p-1)} - pB_{p2}R_1^{-(p+2)}\right] + \frac{1}{\mu_1}\left[A_{p1}(p+1)R_1^{p-1}\right] = \frac{J_p(\theta)}{P_p^1(\cos\theta)} \qquad (55d)$$

$$-\frac{1}{\mu_2}\left[A_{p3}(p+1)R_2^{(p-1)} - pB_{p3}R_2^{-(p+2)}\right] + \frac{1}{\mu_1}\left[A_{p2}(p+1)R_2^{(p-1)} - pB_{p2}R_2^{-(p+2)}\right] = 0 \qquad (55e)$$

$$-\frac{1}{\mu_1}\left[-pB_{p4}R_3^{-(p+2)}\right] + \frac{1}{\mu_2}\left[A_{p3}(p+1)R_3^{(p-1)} - pB_{p3}R_3^{-(p+2)}\right] = 0 \qquad (55f)$$

[Note: $J_\psi(\theta) = J \sum_{p=1}^{\infty} K_p P_p^1(\cos\theta) = \sum_p J_p(\theta)$ where $K_p = 0$ for p even.]

The coefficient A_{p3} in terms of known quantities is expressed as:

$$A_{p3} = \frac{\frac{1}{\mu_1}\left[J_p'(2p+1)R_2^{-(p+2)}\right]}{\left[\frac{1}{\mu_1}(p+1)R_2^{(p-1)} + \frac{1}{\mu_1}[X](p+1)R_2^{-(p+2)} - \frac{1}{\mu_2}(p+1)R_2^{(p-1)} + \frac{1}{\mu_2}\,p[X]R_2^{-(p+2)}\right]} \qquad (56a)$$

$$\text{where } [X] = \frac{-R_3^{2p+1}\left\{1 + \frac{\mu_1}{\mu_2}\left(\frac{p+1}{p}\right)\right\}}{1-\left(\frac{\mu_1}{\mu_2}\right)} \qquad (56b)$$

$$J_p' = \frac{\mu_1 J_p(\theta) R_1^{(p+2)}}{(2p+1) P_p^1(\cos\theta)} \tag{56c}$$

The numerical values for the other five coefficients can be obtained from the following equations

$$B_{p2} = J_p' \tag{57a}$$

$$B_{p3} = A_{p3}\left[X\right] \tag{57b}$$

$$A_{p2} = -J_p' R_2^{-(2p+1)} + A_{p3} + B_{p3} R_2^{-(2p+1)} \tag{57c}$$

$$A_{p1} = A_{p2} + B_{p2} R_1^{-(2p+1)} \tag{57d}$$

$$B_{p4} = A_{p3} R_3^{(2p+1)} + B_{p3} \tag{57e}$$

The coefficients A_{pi} and B_{pi} can be determined from Equations (56) and (57). Equations (51) can now be used to specify the potentials A_I, A_{II}, A_{III}, and A_{IV} in regions I through IV. The normal (B_r) and tangential (B_θ) component of the magnetic induction in regions I through IV can be determined, by using Equations (36) and (40) as:

$$B_{\theta I} = -\sum_{p=1}^{\infty} (p+1) A_{p1} r^{(p-1)} P_p^1 (\cos\theta) \tag{58a}$$

$$B_{rI} = \sum_{p=1}^{\infty} \frac{1}{\sin\theta} \left(A_{p1} r^{(p-1)}\right) \frac{\partial}{\partial\theta} \left[\sin\theta \; P_p^1 (\cos\theta)\right] \tag{58b}$$

$$B_{\theta II} = -\sum_{p=1}^{\infty} \left[A_{p2} (p+1) r^{(p-1)} - pB_{p2} r^{-(p+2)}\right] P_p^1 (\cos\theta) \tag{58c}$$

$$B_{rII} = \sum_{p=1}^{\infty} \frac{1}{\sin\theta} \left[A_{p2} r^{(p-1)} + B_{p2} r^{-(p+2)}\right] \frac{\partial}{\partial\theta} \left[\sin\theta \; P_p^1 (\cos\theta)\right] \tag{58d}$$

$$B_{\theta III} = -\sum_{p=1}^{\infty} \left[A_{p3} (p+1) r^{(p-1)} - pB_{p3} r^{-(p+2)}\right] P_p^1 (\cos\theta) \tag{58e}$$

$$B_{rIII} = \sum_{p=1}^{\infty} \frac{1}{\sin\theta} \left[A_{p3} r^{(p-1)} + B_{p3} r^{-(p+2)}\right] \frac{\partial}{\partial\theta} \left[\sin\theta \; P_p^1 (\cos\theta)\right] \tag{58f}$$

$$B_{\theta IV} = \sum_{p=1}^{\infty} pB_{p4} r^{-(p+2)} P_p^1 (\cos\theta) \tag{58g}$$

$$B_{rIV} = \sum_{p=1}^{\infty} \frac{1}{\sin\theta} \left[B_{p4} r^{-(p+2)}\right] \frac{\partial}{\partial\theta} \left[\sin\theta\, P_p^1 (\cos\theta)\right] \tag{58h}$$

The coefficients A_{pi} ($i = 1,2,3$) and B_{pi} ($i = 2,3,4$), for the vector potential of the present ferromagnetic shell problem, reduce to the coefficients of the potentials in the two regions of a simple current band when the permeability of the shell μ_2 approaches that of the surrounding medium μ_1 (see Appendix C).

Lebedev et al., present the magnetic vector potential due to a dc current I flowing in a filamentary circular loop of radius r_o inside a hollow spherical shell made from material of magnetic permeability μ (in Reference 3 see "The Fourier Method," page 99). The components of the magnetic vector potential were given as:

$$A_r = A_\theta = 0$$

$$A_\psi = A_\psi(r,\theta) = \frac{2\pi I\mu}{c} \sum_{p=0}^{\infty} \frac{(4p+3)^2}{(2p+1)(2p+2)} P^1_{(2p+1)}(0)$$

$$\times \left[\frac{\left(\frac{r_o}{r}\right)^{(2p+2)} P^1_{(2p+1)}(\cos\theta)}{\left[(2p+1)\mu+(2p+2)\right]\left[(2p+2)\mu+(2p+1)\right]} - \left(\frac{R_1}{R_2}\right)^{(4p+3)} (2p+1)(2p+2)(\mu-1)^2 \right]$$

where $r > R_2$; see Figure 7.

$c = 2.998 \times 10^8$ m/sec.

Note: Lebedev's equations are expressed in Gaussian units.

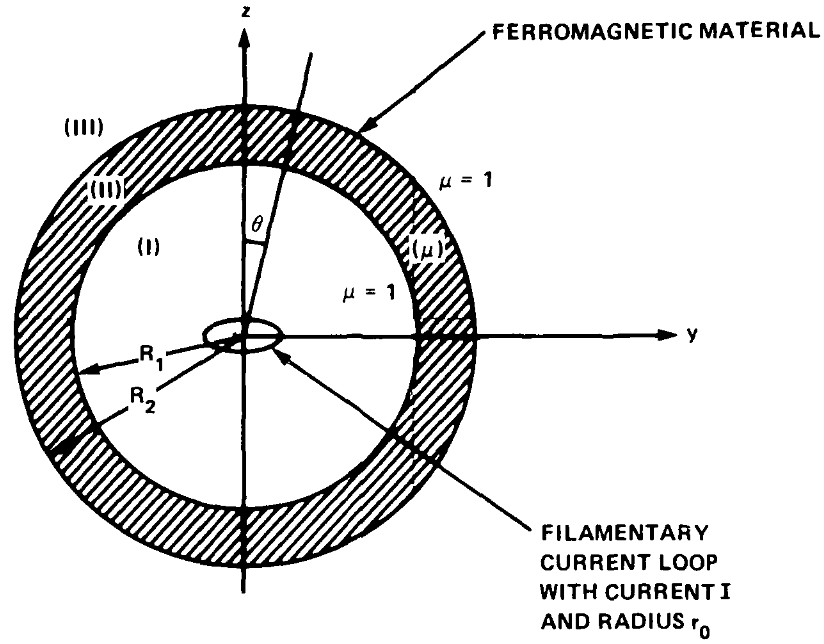


Figure 7 - Ferromagnetic Spherical Shell Surrounding a Filimentary Current Loop

Although we did not derive this solution, a solution for this type of problem could be obtained by allowing the infinitesimally thin current band in the preceding problem to degenerate to a filamentary current loop as was done in Appendix B of Reference 7.

SPHERICAL SHELL SURROUNDED BY AN INFINITESIMALLY THIN SPHERICAL CURRENT BAND

The boundary value problem of a ferromagnetic spherical shell of outer radius R_2, inner radius R_1, and a homogeneous permeability μ_2, surrounded by an infinitesimally thin current band of radius R_3 having a current density $\overline{J}$ was solved in Reference 7. A constant current density was assumed. Figure 8 identifies the four regions of interest. Regions I, III, and IV have a permeability equal to the permeability of vacuum, μ_o, which for convenience will be labeled μ_1. The problem's spherical symmetry suggested that a spherical coordinate system such as that shown in Figure 1 be used in the problem solution.

The form of the potential in each of the regions (I through IV) was determined from the solution of the vector Laplace's equation in each region. These magnetostatic vector potentials in regions I through IV are:

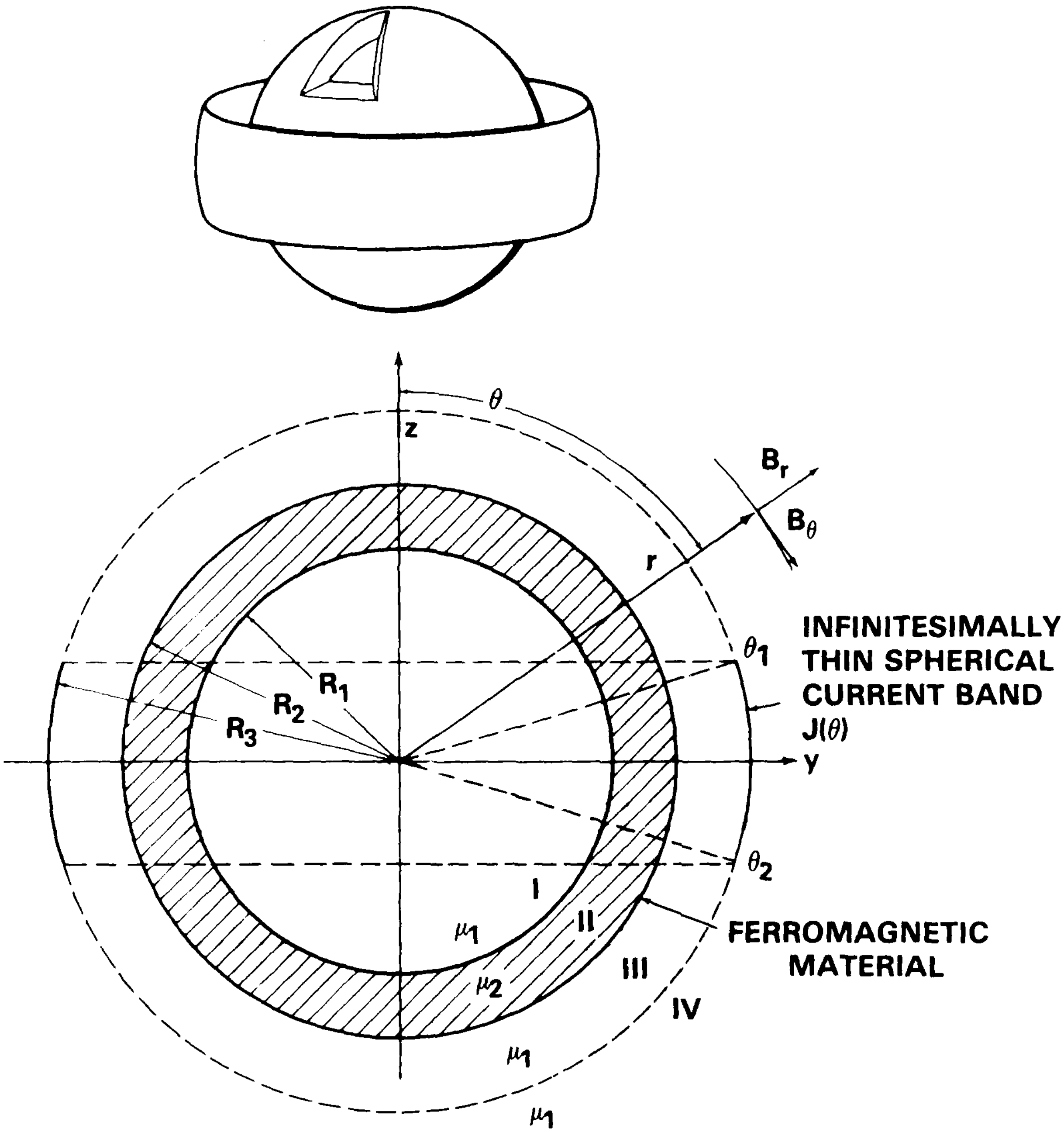


Figure 8 - Ferromagnetic Spherical Shell Surrounded by an Infinitesimally Thin Current Band

$$A_I = A_{\psi I} = \sum_{p=1}^{\infty} \left(A_{p1} r^p\right) P_p^1 (\cos \theta) \qquad (59a)$$

$$A_{II} = A_{\psi II} = \sum_{p=1}^{\infty} \left[A_{p2} r^p + \frac{B_{p2}}{r^{(p+1)}}\right] P_p^1 (\cos \theta) \qquad (59b)$$

$$A_{III} = A_{\psi III} = \sum_{p=1}^{\infty} \left[A_{p3} r^p + \frac{B_{p3}}{r^{(p+1)}}\right] P_p^1 (\cos \theta) \qquad (59c)$$

$$A_{IV} = A_{\psi IV} = \sum_{p=1}^{\infty} \left[\frac{B_{p4}}{r^{(p+1)}}\right] P_p^1 (\cos \theta) \qquad (59d)$$

where for the $A_{\psi I}$ component $B_{p1} = 0$, because at $r = 0$ the potential must be finite, and for the $A_{\psi IV}$ component $A_{p4} = 0$, because as r approaches infinity the potential must remain finite.

At each interface, the basic laws of magnetostatics in Equations (3a) and (3b) reduce to boundary conditions on $\overline{B}$ and $\overline{H}$ that can be used to evaluate the six constants in Equations (59). From Equation (18a), the normal component of $\overline{B}$ across each boundary must be continuous, i.e., $(\overline{B}_2-\overline{B}_1) \cdot \overline{n}_{12} = 0$ where the quantity $\overline{n}_{12}$ is the unit outward normal to the surface. This provides the following boundary conditions which must be satisfied by the solution in Equations (59) for each region.

$$B_{rI} = B_{rII} \quad \text{at } r = R_1 \qquad (60a)$$

$$B_{rII} = B_{rIII} \quad \text{at } r = R_2 \qquad (60b)$$

$$B_{rIII} = B_{rIV} \quad \text{at } r = R_3 \qquad (60c)$$

The normal component of the magnetic field B_r is expressed in terms of the vector potential as

$$B_r = (\overline{\nabla} x \overline{A})_r \tag{61a}$$

$$B_r = \frac{1}{r \sin\theta} \frac{\partial}{\partial\theta} (\sin\theta \; A_\psi) \tag{61b}$$

where $\overline{B} = \overline{\nabla} \times \overline{A} = \dfrac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{e}_r & \hat{e}_\theta r & \hat{e}_\psi r \sin\theta \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial\psi} \\ 0 & 0 & A_\psi r \sin\theta \end{vmatrix}$

However, because the vector potentials in each region are functions of P_p^1 (cos θ) we can simplify Equations (60) to constraints on A_ψ

$$A_I = A_{II} \quad \text{at } r = R_1 \tag{62a}$$

$$A_{II} = A_{III} \quad \text{at } r = R_2 \tag{62b}$$

$$A_{III} = A_{IV} \quad \text{at } r = R_3 \tag{62c}$$

The second set of boundary conditions is obtained from Equation (18b). The tangential component of $\overline{H}$ across each boundary must satisfy the relationship

$$\overline{n}_{12} \times (\overline{H}_2 - \overline{H}_1) = \overline{J}_s \tag{63}$$

where $\overline{J}_s$ (which equals $\overline{J}(\theta)$) is the real surface current density in the limit of vanishing width between the two regions. Using the relationship $\overline{B} = \mu\overline{H}$, Equation (63) may be expressed as

$$\frac{B_{\theta 2}}{\mu_2} - \frac{B_{\theta 1}}{\mu_1} = J(\theta) \tag{64}$$

Referring to the curl in Equation (61), we can write B_θ as

$$B_\theta = (\overline{\nabla}\times\overline{A})_\theta = -\frac{1}{r}\frac{\partial}{\partial r}\left[r\ A_\psi\right] \tag{65}$$

$$-\frac{1}{\mu_2}\frac{1}{r}\frac{\partial}{\partial r}(rA_{II}) + \frac{1}{\mu_1}\frac{1}{r}\frac{\partial}{\partial r}(rA_I) = 0 \quad \text{at } r = R_1 \tag{66a}$$

$$-\frac{1}{\mu_1}\frac{1}{r}\frac{\partial}{\partial r}(rA_{III}) + \frac{1}{\mu_2}\frac{1}{r}\frac{\partial}{\partial r}(rA_{II}) = 0 \quad \text{at } r = R_2 \tag{66b}$$

$$-\frac{1}{\mu_1}\frac{1}{r}\frac{\partial}{\partial r}(rA_{IV}) + \frac{1}{\mu_1}\frac{1}{r}\frac{\partial}{\partial r}(rA_{III}) = J(\theta) \quad \text{at } r = R_3 \tag{66c}$$

The general expressions for the potentials in each region (Equations (59)) are then substituted into the boundary conditions (Equations (62) and (66)) and solved for the constants A_{pi} and B_{pi}. There are six algebraic equations with six unknowns and the potential in each region can then be specifically determined. The six boundary value equations that must be solved for the coefficients are given next (where the index p is odd only and understood to take on values from 1 to ∞). It is noted that the current $J_\psi(\theta)$ must be expanded into a set of associated Legendre functions in order to evaluate the constants A_{pi} and B_{pi}. The detailed expansion is derived in the section of Reference 7 entitled "Expansion of the Current ($J_\psi(\theta)$) in Associated Legendre polynomials."

$$A_{p1}R_1{}^{p} = \left[A_{p2}R_1{}^{p}+B_{p2}R_1{}^{-(p+1)}\right] \tag{67a}$$

$$\left[A_{p2}R_2{}^{p} +B_{p2}R_2{}^{-(p+1)}\right] = \left[A_{p3}R_2{}^{p}+B_{p3}R_2{}^{-(p+1)}\right] \tag{67b}$$

$$\left[A_{p3}R_3{}^{p}+B_{p3}R_3{}^{-(p+1)}\right] = B_{p4}\left[R_3{}^{-(p+1)}\right] \tag{67c}$$

$$-\frac{1}{\mu_2}\left[A_{p2}(p+1)R_1{}^{(p-1)} - pB_{p2}R_1{}^{-(p+2)}\right] + \frac{1}{\mu_1}\left[A_{p1}(p+1)R_1{}^{(p-1)}\right] = 0 \tag{67d}$$

$$-\frac{1}{\mu_1}\left[A_{p3}(p+1)R_2{}^{(p-1)} - pB_{p3}R_2{}^{-(p+2)}\right] + \frac{1}{\mu_2}\left[A_{p2}(p+1)R_2{}^{(p-1)} - pB_{p2}R_2{}^{-(p+2)}\right] = 0 \tag{67e}$$

$$-\frac{1}{\mu_1}\left[- pB_{p4}R_3^{-(p+2)}\right] + \frac{1}{\mu_1}\left[A_{p3}(p+1)R_3^{p-1} - pB_{p3}R_3^{-(p+2)}\right] = J_p(\theta)\Big/P_p^1(\cos\theta) \tag{67f}$$

The solution of these equations to obtain B_{p3} in terms of known quantities is performed in Appendix A of Reference 7. In summary:

$$B_{p3} = \frac{-\frac{1}{\mu_2} J''_p(\theta)\left([X](p+1)R_2^{(p-1)} - pR_2^{-(p+2)}\right) + \frac{1}{\mu_1} J'_p(\theta)(p+1)R_2^{(p-1)}}{\left[\left(\frac{1}{\mu_1} pR_2^{-(p+2)}\right) + \frac{1}{\mu_2}\left([Z][X](p+1)R_2^{(p-1)}\right) - \frac{1}{\mu_2}\left([Z]pR_2^{-(p+2)}\right)\right]} \tag{68a}$$

$$[X] = \frac{-R_1^{-(2p+1)}\left[1+\left(\frac{p}{p+1}\right)\frac{\mu_1}{\mu_2}\right]}{\left(1-\frac{\mu_1}{\mu_2}\right)} \tag{68b}$$

$$[Z] = \frac{R_2^{-(p+1)}}{\left([X]R_2^{p}+R_2^{-(p+1)}\right)} \tag{68c}$$

$$J''_p(\theta) = \frac{J'_p(\theta)R_2^{p}}{\left([X]R_2^{p}+R_2^{-(p+1)}\right)} \tag{68d}$$

$$J'_p(\theta) = \frac{\mu_1 J_p(\theta)}{P_p^1(\cos\theta)R_3^{(p-1)}(2p+1)} \tag{68e}$$

The numerical values for the other five coefficients can be obtained from the following equations:

$$B_{p2} = B_{p3} \left[Z\right] + J_p'' (\theta) \tag{69a}$$

$$A_{p3} = J_p' (\theta) \tag{69b}$$

$$A_{p2} = \left[X\right] B_{p2} \tag{69c}$$

$$A_{p1} = A_{p2} + B_{p2} R_1^{-(2p+1)} \tag{69d}$$

$$B_{p4} = A_{p3} R_3^{(2p+1)} + B_{p3} \tag{69e}$$

Because the coefficients A_{pi} and B_{pi} can be determined from Equations (68) and (69), Equations (59) can now be used to completely specify the potentials A_I, A_{II}, A_{III}, and A_{IV} in regions I through IV. The normal (B_r) and tangential (B_θ) components of the magnetic induction in regions I through IV can be determined by using Equations (61) and (65) as:

$$B_{\theta I} = - \sum_{p=1}^{\infty} (p+1) \left(A_{p1} r^{(p-1)}\right) P_p^1 (\cos \theta) \tag{70a}$$

$$B_{rI} = \frac{1}{\sin \theta} \sum_{p=1}^{\infty} \left[A_{p1} r^{(p-1)}\right] \frac{\partial}{\partial \theta} \left(\sin \theta \; P_p^1 (\cos \theta)\right) \tag{70b}$$

$$B_{\theta II} = - \sum_{p=1}^{\infty} \left[A_{p2}(p+1)r^{(p-1)} - pB_{p2}r^{-(p+2)}\right] P_p^1 (\cos \theta) \tag{70c}$$

$$B_{rII} = \sum_{p=1}^{\infty} \frac{1}{\sin \theta} \left[A_{p2}r^{(p-1)} + B_{p2}r^{-(p+2)}\right] \frac{\partial}{\partial\theta} \left(\sin \theta \; P_p^1 (\cos \theta)\right) \tag{70d}$$

$$B_{\theta III} = - \sum_{p=1}^{\infty} \left[A_{p3}(p+1)r^{(p-1)} - pB_{p3}r^{-(p+2)}\right] P_p^1 (\cos \theta) \tag{70e}$$

$$B_{rIII} = \sum_{p=1}^{\infty} \frac{1}{\sin \theta} \left[A_{p3}r^{(p-1)} + B_{p3}r^{-(p+2)}\right] \frac{\partial}{\partial\theta} \left(\sin \theta \; P_p^1 (\cos \theta)\right) \tag{70f}$$

$$B_{\theta IV} = \sum_{p=1}^{\infty} \left(pB_{p4}r^{-(p+2)}\right) P_p^1 (\cos \theta) \tag{70g}$$

$$B_{rIV} = \sum_{p=1}^{\infty} \frac{1}{\sin \theta} \left[B_{p4}r^{-(p+2)}\right] \frac{\partial}{\partial\theta} \left(\sin \theta \; P_p^1 (\cos \theta)\right) \tag{70h}$$

In Appendix C of Reference 7, the coefficients A_{pi} $(i = 1,2,3)$ and B_{pi} $(i = 2,3,4)$ for the vector potentials for this ferromagnetic shell problem were shown to reduce to the potentials in the two regions of the simple current band problem when the permeability of the ferromagnetic shell μ_2 approaches that of the surrounding medium μ_1. This showed that the solutions of that ferromagnetic current problem had the correct mathematical form.

SPHERICAL SHELL WITH INTERNAL AND EXTERNAL INFINITESIMALLY THIN SPHERICAL CURRENT BANDS

We now proceed to solve the boundary value problem of a ferromagnetic spherical shell of outer radius R_3, inner radius R_2, having a homogeneous permeability μ_2, surrounded by an infinitesimally thin current band of radius R_4 having a current density $\bar{J}_2$ and surrounding an infinitesimally thin current band of radius R_1 having a current density $\bar{J}_1$. A constant current density is assumed for both bands. Figure 9 identifies the five regions of interest. Regions I, II, IV, and V have a permeability equal to the permeability of vacuum, μ_o, which for convenience will be labeled μ_1. The problem's spherical symmetry suggests that a spherical coordinate system such as that shown in Figure 1 be used in the problem solution. The governing differential equation for $\bar{A}$ when homogeneous and linear materials are considered is, from Equation (17),

$$\nabla^2 \bar{A} = -\mu\bar{J} \tag{71}$$

From Equation (22), we see that the elemental vector potential $\overline{dA}$ due to a current element $\bar{J}dv$ is in the same direction as $\bar{J}$. It is well known from this that the lines of the magnetic vector potential $\bar{A}$ are circles centered about the coil or loop axis. The magnitude of $\bar{A}$ along such a circle is constant, which means that $\bar{A}$ is a function of the spherical coordinates r and θ only. Therefore, we know in advance for this problem that A_ψ is the only component of $\bar{A}$ existing at the field point. The infinitesimally thin bands of current shown in Figure 9 have only an azimuthal or ψ component, which is a function of r and θ, and lie on the boundaries between regions I and II (i.e., $r = R_1$) and between regions IV and V (i.e., $r = R_4$). These currents can be expressed as:

$$\bar{J}_1 = \begin{cases} 0 \quad , & \text{if } \theta < \theta_1 \quad \text{or } \theta > \theta_2 \\ \hat{e}_\psi J_\psi(\theta), & \text{if} \quad \theta_1 \le \theta \le \theta_2 \end{cases} \tag{72}$$

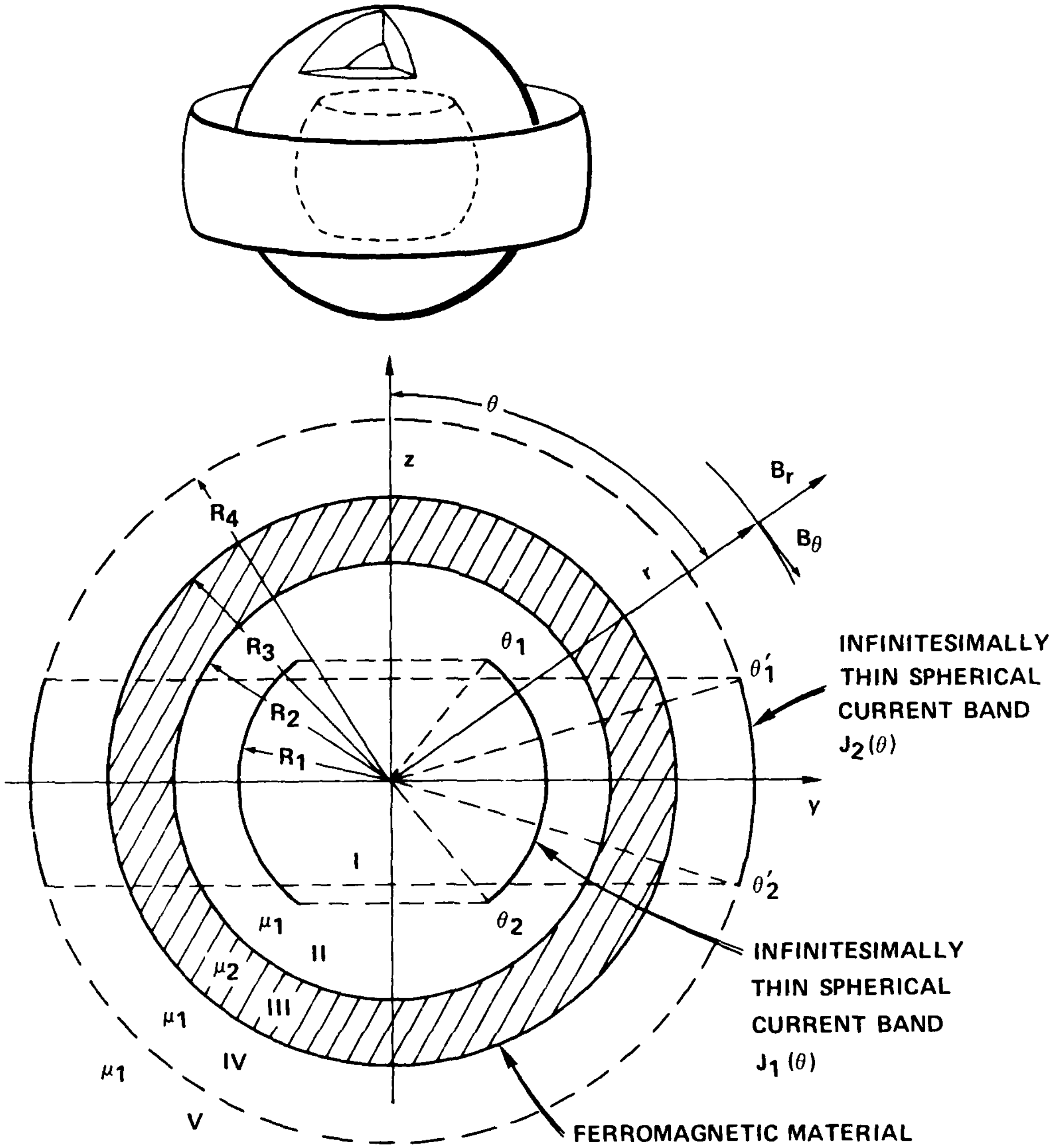


Figure 9 - Ferromagnetic Spherical Shell with Internal and External Infinitesimally Thin Current Bands

$$
\overline{J}_2 = \begin{cases} 0 & , \text{ if } \theta < \theta_1' \text{ or } \theta > \theta_2' \\ \hat{e}_\psi J_\psi(\theta) & , \text{ if } \theta_1' \leq \theta \leq \theta_2' \end{cases} \tag{73}
$$

Therefore, Equation (71) has only an azimuthal component and can be expressed as:

$$
\text{✡} A_\psi = \text{✡} A_\psi (r,\theta) = 0 \text{ (in regions I through V)} \tag{74}
$$

When the vector Laplacian ✡ is expanded in spherical coordinates, Equation (74) can be written as

$$
\frac{\partial^2 A_\psi}{\partial r^2} + \frac{2}{r}\frac{\partial A_\psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 A_\psi}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial A_\psi}{\partial \theta} - \frac{A_\psi}{r^2 \sin^2\theta} = 0 \quad \text{in regions I through IV} \tag{75}
$$

To solve Equation (75), we follow the procedure given on page 17. The general solution of Equation (75) in regions I through IV may be formed from the product of the solutions in Equation (31) which yields

$$
\begin{aligned}
A_\psi = R(r)\Theta(\theta) &= \sum_{p=1}^{\infty} R_p(r)\Theta_p(\theta) \\
&= \sum_{p=1}^{\infty} \left(A_p' r^p + \frac{B_p'}{r^{(p+1)}}\right)\left(C_p P_p^1 (\cos\theta) + D_p Q_p^1 (\cos\theta)\right)
\end{aligned} \tag{76}
$$

In the spherical case, associated Legendre functions of the second kind are infinite at $\cos\theta = \pm 1$, and thus cannot be included when the region under consideration includes the symmetry axis. Therefore, the constant D_p must be set equal to zero, and Equation (76) reduces to

$$A_\psi = \sum_{p=1}^{\infty} \left(A_p r^p + \frac{B_p}{r^{(p+1)}} \right) P_p^1 (\cos\theta) \tag{77}$$

where $A_p = A_p' C_p$, and $B_p = B_p' C_p$.

The form of the potential in each of the regions (I through V) is determined from Equation (77). These magnetostatic vector potentials in regions I through V are:

$$A_I = A_{\psi I} = \sum_{p=1}^{\infty} \left[A_{p1} r^p \right] P_p^1 (\cos\theta) \tag{78a}$$

$$A_{II} = A_{\psi II} = \sum_{p=1}^{\infty} \left[A_{p2} r^p + \frac{B_{p2}}{r^{(p+1)}} \right] P_p^1 (\cos\theta) \tag{78b}$$

$$A_{III} = A_{\psi III} = \sum_{p=1}^{\infty} \left[A_{p3} r^p + \frac{B_{p3}}{r^{(p+1)}} \right] P_p^1 (\cos\theta) \tag{78c}$$

$$A_{IV} = A_{\psi IV} = \sum_{p=1}^{\infty} \left[A_{p4} r^p + \frac{B_{p4}}{r^{(p+1)}} \right] P_p^1 (\cos\theta) \tag{78d}$$

$$A_V = A_{\psi V} = \sum_{p=1}^{\infty} \left[\frac{B_{p5}}{r^{(p+1)}} \right] P_p^1 (\cos\theta) \tag{78e}$$

where, for the $A_{\psi I}$ component, $B_{p1} = 0$, because at $r = 0$ the potential must be finite and, for the $A_{\psi V}$ component, $A_{p5} = 0$, because as r approaches infinity the potential must remain finite.

At each interface, the basic laws of magnetostatics in Equation (2a) reduce to boundary conditions on $\overline{B}$ and $\overline{H}$ that can be used to evaluate the eight constants in Equations (78). From Equation (18a), the normal component of $\overline{B}$ across each boundary must be continuous, i.e., $(\overline{B}_2 - \overline{B}_1) \cdot \overline{n}_{12} = 0$ where the quantity $\overline{n}_{12}$ is the unit outward normal to the surface. This provides the following boundary conditions which must be satisfied by the solution in Equations (78) for each region.

$$\begin{aligned} B_{rI} &= B_{rII} \quad \text{at } r = R_1 \\ B_{rII} &= B_{rIII} \quad \text{at } r = R_2 \\ B_{rIII} &= B_{rIV} \quad \text{at } r = R_3 \\ B_{rIV} &= B_{rV} \quad \text{at } r = R_4 \end{aligned} \tag{79}$$

The normal component of the magnetic field B_r is expressed in terms of the vector potential as

$$\begin{aligned} B_r &= (\overline{\nabla} x \overline{A})_r \\ B_r &= \frac{1}{r \sin\theta} \frac{\partial}{\partial\theta} (\sin\theta \, A_\psi) \end{aligned} \tag{80}$$

$$\text{where}\quad \overline{B} = \overline{\nabla} \times \overline{A} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \widehat{e}_r & \widehat{e}_\theta r & \widehat{e}_\psi r \sin\theta \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \psi} \\ 0 & 0 & A_\psi r \sin\theta \end{vmatrix}$$

However, because the vector potentials in each region are functions of P_p^1 (cos θ) we can simplify Equation (79) to constraints on A_ψ

$$A_I = A_{II} \quad \text{at } r = R_1 \tag{81a}$$

$$A_{II} = A_{III} \quad \text{at } r = R_2 \tag{81b}$$

$$A_{III} = A_{IV} \quad \text{at } r = R_3 \tag{81c}$$

$$A_{IV} = A_V \quad \text{at } r = R_4 \tag{81d}$$

The second set of boundary conditions is obtained from Equation (18b). The tangential component of $\overline{H}$ across each boundary must satisfy the relationship

$$\overline{n}_{12} \times (\overline{H}_2 - \overline{H}_1) = \overline{J}_s \tag{82}$$

where $\overline{J}_s$ (which equals $\overline{J}(\theta)$) is the real surface current density in the limit of vanishing width between the two regions. Using the relationship $\overline{B} = \mu\overline{H}$, Equation (82) may be expressed as

$$\frac{B_{\theta 2}}{\mu_2} - \frac{B_{\theta 1}}{\mu_1} = J(\theta) \tag{83}$$

Referring to the curl in Equation (80), we can write B_θ as

$$B_\theta = (\overline{\nabla} x \overline{A})_\theta = - \frac{1}{r} \frac{\partial}{\partial r} \left[rA_\psi \right] \tag{84}$$

From Equations (82) through (84) the tangential components in regions I through V must satisfy the relationships:

$$- \frac{1}{\mu_1} \frac{1}{r} \frac{\partial}{\partial r} (rA_{II}) + \frac{1}{\mu_1} \frac{1}{r} \frac{\partial}{\partial r} (rA_I) = J_1(\theta) \text{ at } r = R_1 \tag{85a}$$

$$- \frac{1}{\mu_2} \frac{1}{r} \frac{\partial}{\partial r} (rA_{III}) + \frac{1}{\mu_1} \frac{1}{r} \frac{\partial}{\partial r} (rA_{II}) = 0 \text{ at } r = R_2 \tag{85b}$$

$$- \frac{1}{\mu_1} \frac{1}{r} \frac{\partial}{\partial r} (rA_{IV}) + \frac{1}{\mu_2} \frac{1}{r} \frac{\partial}{\partial r} (rA_{III}) = 0 \text{ at } r = R_3 \tag{85c}$$

$$- \frac{1}{\mu_1} \frac{1}{r} \frac{\partial}{\partial r} (rA_V) + \frac{1}{\mu_1} \frac{1}{r} \frac{\partial}{\partial r} (rA_{IV}) = J_2(\theta) \text{ at } r = R_4 \tag{85d}$$

The general expressions for the potentials in each region (Equations (78)) are then substituted into the boundary conditions (Equations (81) and (85)) and solved for the constants A_{pi} and B_{pi}. There are eight algebraic equations with eight unknowns, and the potential in each region can then be specifically determined. The eight boundary value equations that must be solved for the coefficients are given next (where the index p is odd only and understood to take on values from 1 to ∞). It is noted that the current $J_\psi(\theta)$ must be expanded into a set of associated Legendre functions in order to evaluate the constants A_{pi} and B_{pi}. The detailed expansion is derived in the section of Reference 7 entitled "Expansion of the Current ($J_\psi(\theta)$) in Associated Legendre Polynominals."

$$A_{p1}R_1^p = \left[A_{p2}R_1^p + B_{p2}R_1^{-(p+1)}\right] \tag{86a}$$

$$\left[A_{p2}R_2^p + B_{p2}R_2^{-(p+1)}\right] = \left[A_{p3}R_2^p + B_{p3}R_2^{-(p+1)}\right] \tag{86b}$$

$$\left[A_{p3}R_3^p + B_{p3}R_3^{-(p+1)}\right] = \left[A_{p4}R_3^p + B_{p4}R_3^{-(p+1)}\right] \tag{86c}$$

$$\left[A_{p4}R_4^p + B_{p4}R_4^{-(p+1)}\right] = \left[B_{p5}R_4^{-(p+1)}\right] \tag{86d}$$

$$-\frac{1}{\mu_1}\left[A_{p2}(p+1)R_1^{(p-1)} - pB_{p2}R_1^{-(p+2)}\right] + \frac{1}{\mu_1}\left[(p+1)A_{p1}R_1^{(p-1)}\right] = \frac{J_{p1}(\theta)}{P_p^1(\cos\theta)} \tag{86e}$$

$$-\frac{1}{\mu_2}\left[A_{p3}(p+1)R_2^{(p-1)} - pB_{p3}R_2^{-(p+2)}\right] + \frac{1}{\mu_1}\left[A_{p2}(p+1)R_2^{(p-1)} - pB_{p2}R_2^{-(p+2)}\right] = 0 \tag{86f}$$

$$-\frac{1}{\mu_1}\left[A_{p4}(p+1)R_3^{(p-1)} - pB_{p4}R_3^{-(p+2)}\right] + \frac{1}{\mu_2}\left[A_{p3}(p+1)R_3^{(p-1)} - pB_{p3}R_3^{-(p+2)}\right] = 0 \tag{86g}$$

$$+\frac{1}{\mu_1}\left[pR_4^{-(p+2)}B_{p5}\right]+\frac{1}{\mu_1}\left[A_{p4}(p+1)R_4^{(p-1)}-pB_{p4}R_4^{-(p+2)}\right]=\frac{J_{p2}(\theta)}{P_p^1(\cos\theta)} \tag{86h}$$

The solution of these equations is performed in Appendix D. In summary:

$$A_{p4}=\frac{\mu_1 J_{p2}(\theta)}{(2p+1)R_4^{(p-1)}\ P_p^1\ (\cos\theta\ )}\equiv[X] \tag{87a}$$

$$B_{p2}=\frac{\mu_1 J_{p1}(\theta)}{(2p+1)R_1^{-(p+2)}P_p^1\ (\cos\theta)}\equiv[Y] \tag{87b}$$

$$A_{p2}=\frac{A_{p3}\left(\frac{(2p+1)}{p}\right)R_2^{(2p+1)}+\left(\frac{\mu_2}{\mu_1}-1\right)B_{p2}}{\left[1+\frac{\mu_2}{\mu_1}\left(\frac{(p+1)}{p}\right)\right]R_2^{(2p+1)}} \tag{87c}$$

$$A_{p3}=B_{p3}[W]+[X][S]\equiv[Z] \tag{87d}$$

$$B_{p3}=\frac{R_2^p[X][S]-R_2^p[X][S][T]-[Y][A]R_2^p-[Y]R_2^{-(p+1)}}{R_2^p\ [W][T]-R_2^{-(p+1)}-R_2^p\ [W]} \tag{87e}$$

$$B_{p4} = A_{p3}R_3^{(2p+1)} + B_{p3} - A_{p4}R_3^{(2p+1)} \tag{87f}$$

$$B_{p5} = - A_{p4}\left(\frac{(p+1)}{p}\right) R_4^{(2p+1)} + B_{p4} + \frac{\mu_1 J_{p2}(\theta)}{P_p^1(\cos\theta)p\ R_4^{-(p+2)}} \tag{87g}$$

$$A_{p1} = A_{p2} + B_{p2}R_1^{-(2p+1)} \tag{87h}$$

where

$$[T] = \frac{\left(\frac{(2p+1)}{p}\right) R_2^{(2p+1)}}{\left\{\left[1 + \frac{\mu_2}{\mu_1}\left(\frac{(p+1)}{p}\right)\right] R_2^{(2p+1)}\right\}} \tag{87i}$$

$$[A] = \frac{\left(\frac{\mu_2}{\mu_1} - 1\right)}{\left\{\left[1 + \frac{\mu_2}{\mu_1}\left(\frac{(p+1)}{p}\right)\right] R_2^{(2p+1)}\right\}} \tag{87j}$$

$$[W] = \frac{\left(\frac{\mu_1}{\mu_2} - 1\right)}{\left[\left(\frac{\mu_1}{\mu_2}\right)\left(\frac{(p+1)}{p}\right) + 1\right]R_3^{(2p+1)}} \tag{87k}$$

$$[S] = \frac{(2p+1)}{p\left[\left(\frac{\mu_1}{\mu_2}\right)\left(\frac{p+1}{p}\right) + 1\right]} \tag{87ℓ}$$

Because the coefficients A_{pi} and B_{pi} can be determined from Equations (86) and (87), Equations (78) can now be used to completely specify the potentials A_I, A_{II}, A_{III}, and A_{IV} in regions I through IV. Then the normal (B_r) and tangential (B_θ) components of the magnetic induction in regions I through IV can be determined by using Equations (80) and (84), to be:

$$B_{\theta I} = - \sum_{p=1}^{\infty} \left[(p+1)A_{p1}r^{(p-1)}\right]P_p^1(\cos\theta) \tag{88a}$$

$$B_{rI} = \frac{1}{\sin\theta} \sum_{p=1}^{\infty} \left(A_{p1}r^{(p-1)}\right) \frac{\partial}{\partial\theta}\left[\sin\theta \; P_p^1(\cos\theta)\right] \tag{88b}$$

$$B_{\theta II} = \sum_{p=1}^{\infty} \left[-(p+1)A_{p2}r^{(p-1)} + pB_{p2}r^{-(p+2)}\right] P_p^1(\cos\theta) \tag{88c}$$

$$B_{rII} = \frac{1}{\sin\theta} \sum_{p=1}^{\infty} \left[A_{p2}r^{(p-1)} + \frac{B_{p2}}{r^{(p+2)}}\right] \frac{\partial}{\partial\theta}\left[\sin\theta \; P_p^1(\cos\theta)\right] \tag{88d}$$

$$B_{\theta III} = \sum_{p=1}^{\infty} \left[-(p+1)A_{p3}r^{(p-1)} + pB_{p3}r^{-(p+2)}\right] P_p^1(\cos\theta) \tag{88e}$$

$$B_{rIII} = \frac{1}{\sin\theta} \sum_{p=1}^{\infty} \left[A_{p3}r^{(p-1)} + \frac{B_{p3}}{r^{(p+2)}}\right] \frac{\partial}{\partial\theta}\left[\sin\theta \; P_p^1(\cos\theta)\right] \tag{88f}$$

$$B_{\theta IV} = \sum_{p=1}^{\infty} \left[-(p+1)A_{p4}r^{(p-1)} + pB_{p4}r^{-(p+2)}\right] P_p^1(\cos\theta) \tag{88g}$$

$$B_{rIV} = \frac{1}{\sin\theta} \sum_{p=1}^{\infty} \left[A_{p4}r^{(p-1)} + \frac{B_{p4}}{r^{(p+2)}}\right] \frac{\partial}{\partial\theta}\left[\sin\theta\, P_p^1(\cos\theta)\right] \tag{88h}$$

$$B_{\theta V} = \sum_{p=1}^{\infty} \left[pB_{p5}r^{-(p+2)}\right] P_p^1(\cos\theta) \tag{88i}$$

$$B_{rV} = \frac{1}{\sin\theta} \sum_{p=1}^{\infty} \left[B_{p5}r^{-(p+2)}\right] \frac{\partial}{\partial\theta}\left[\sin\theta\, P_p^1(\cos\theta)\right] \tag{88j}$$

In Appendix D, the coefficients A_{pi} $(i = 1,2,3,4)$ and B_{pi} $(i = 2,3,4,5)$ for the vector potentials for the present ferromagnetic shell problem reduce to the potentials in the two regions of the simple current band problem when the permeability of the ferromagnetic shell μ_2 approaches that of the surrounding medium μ_1. This shows that the solutions of the above ferromagnetic current problem have the correct mathematical form.

SOLUTIONS FOR PROLATE SPHEROIDAL BODIES

SOLID PROLATE SPHEROID OR PROLATE SPHEROIDAL SHELL IN A UNIFORM FIELD OF ARBITRARY DIRECTION

Important problems relating to determining the magnetic induction for spheroidal ferromagnetic bodies in an external magnetic field have not been widely reported on in the literature or in text books. The solutions for these types of boundary value problems can be obtained by using a procedure similar to that used for spherical bodies[3,4,5] by determining the solution to Laplace's equation for the magnetic scalar potential. Constant external field problems, solid and shell, were solved and programmed on the digital computer by Nixon of the Center, for the case of an arbitrarily oriented external magnetic field.[6] The solutions found in Reference 6 were presented in Cartesian coordinates. The problem of, for instance, finding the magnetic induction for an infinitesimally thin current band surrounding a spheroidal shell[8] can be generalized to include an external magnetic field. Linear superposition may be applied to find the solution in this case. Therefore, in Appendix E the Cartesian expressions[6] were converted to prolate spheroidal coordinates to be compatible with other problem solutions in this section of the report.

SOLID PROLATE SPHEROID SURROUNDED BY AN INFINITESIMALLY THIN SPHEROIDAL CURRENT BAND

For the case of the prolate spheroidal bodies[8,9] the equations given in the Basic Equations section of this text apply. The governing differential equation for $\overline{A}$ when homogenous and linear materials are considered is from Equation (17).

$$✡\, \overline{A} = -\mu \overline{J} \tag{89}$$

where the general expression in prolate spheroidal coordinates for a current density is

$$\overline{J} = J_\eta\, \hat{e}_\eta + J_\theta\, \hat{e}_\theta + J_\psi\, \hat{e}_\psi \tag{90}$$

As previously discussed (see page 16), because the current has only an azimuthal or ψ component, $\overline{A}$ has only an A_ψ component. For the spheroidal problems considered in this report, the current band is assumed to be infinitesimally thin and the governing differential equation for $\overline{A}$ in each region can be expressed as

$$\frac{\partial}{\partial \eta}\left[\frac{1}{\sinh \eta}\,\frac{\partial(\sinh \eta\, A_\psi)}{\partial \eta}\right] + \frac{\partial}{\partial \theta}\left[\frac{1}{\sin \theta}\,\frac{\partial(\sin \theta\, A_\psi)}{\partial \theta}\right] = 0 \qquad (91)$$

Using the method of separation of variables, the solution to Equation (91) is

$$A_\psi = \sum_{p=1}^{\infty} \left[A\, P_p^1(\cosh \eta) + B\, Q_p^1(\cosh \eta)\right]$$

$$\times \left[A'\; P_p^1(\cos \theta) + B'\; Q_p^1(\cos \theta)\right] \qquad (92)$$

where P_p^m and Q_p^m are the associated Legendre functions of the first and second kind, respectively.

For the prolate spheroidal system, the associated Legendre functions of the second kind are infinite at $\cos \theta = \pm 1$, and as such cannot be included in a general solution for a given region which includes $\theta = 0$, or $\theta = \pi$. Therefore, in our case, the constant B' is set equal to zero. Equation (92) reduces to

$$A_\psi = \sum_{p=1}^{\infty} \left[k_1 P_p^1(\cosh \eta) + k_2 Q_p^1(\cosh \eta)\right] P_p^1\,(\cos \theta) \qquad (93)$$

where k_1 and k_2 are constants ($k_1 = AA'$, $k_2 = BA'$). When the substitutions $\xi = \cosh \eta$ and $\nu = \cos \theta$ are made in Equation (93), A_ψ can be expressed as

$$A_\psi = \sum_{p=1}^{\infty} \left[k_1 P_p^1(\xi) + k_2 Q_p^1(\xi)\right] P_p^1(\nu) \qquad (94)$$

This is the general form of the psi (ψ) component of the vector potential that will be used to determine the potentials A_ψ in each region.

The problem of a solid ferromagnetic prolate spheroid surrounded by an infinitesimally thin prolate spheroidal current band shown in Figure 10 was solved by Purczynski.[10]

In this case, the permeability of the solid spheroid is μ_2 and the boundary of the body is determined by $\eta = \eta_1$ = constant. The permeability μ_o of a vacuum that is external to the spheroid is denoted by μ_1. The current band which lies in the boundary between regions II and III is denoted by $\eta = \eta_2$ = constant, and the constant current density flowing in the band is $\overline{J}$.

For completeness, Purczynski's work[10] is presented in this text in our notation. The form of the components of the vector potential A_ψ in regions I through III is determined from Equation (94). These magnetostatic vector potentials in regions I, II, and III are:

$$A_{\psi I} = \sum_{p=1}^{\infty} \left[A_p P_p^1(\xi)\right] P_p^1(\nu) \tag{95a}$$

$$A_{\psi II} = \sum_{p=1}^{\infty} \left[B_p P_p^1(\xi) + D_p Q_p^1(\xi)\right] P_p^1(\nu) \tag{95b}$$

$$A_{\psi III} = \sum_{p=1}^{\infty} \left[E_p Q_p^1(\xi)\right] P_p^1(\nu) \tag{95c}$$

where $\xi = \cosh \eta$ and $\nu = \cos \theta$.

We note two constants were set equal to zero because the potential must be finite in each of the regions I through III and approach zero as $\xi \to \infty$ in region III.

The remaining constants are determined from the boundary conditions

$$B_{\eta I} = B_{\eta II} \quad \text{at } \eta = \eta_1 \tag{96a}$$

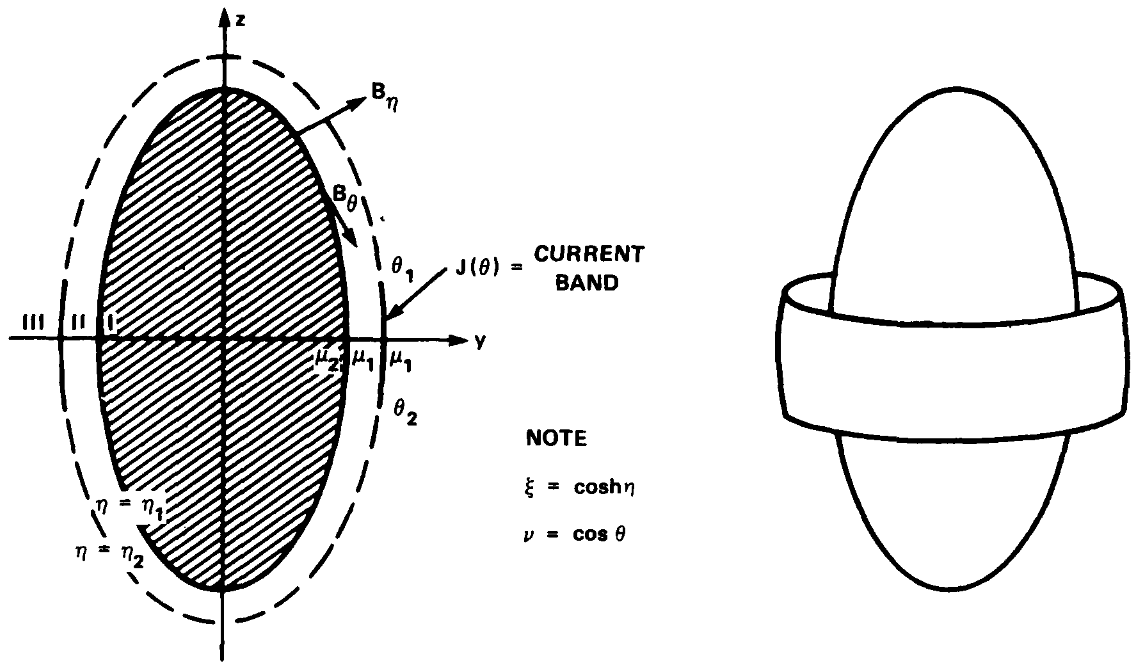


Figure 10 - Ferromagnetic Prolate Spheroidal Solid Surrounded by an Infinitesimally Thin Current Band

$$B_{\eta II} = B_{\eta III} \text{ at } \eta = \eta_2 \tag{96b}$$

$$\frac{B_{\theta III}}{\mu_1} - \frac{B_{\theta II}}{\mu_1} = J_\psi(\theta) \text{ at } \eta = \eta_2 \tag{96c}$$

$$\frac{B_{\theta II}}{\mu_1} - \frac{B_{\theta I}}{\mu_2} = 0 \quad \text{at } \eta = \eta_1 \tag{96d}$$

We note that Equations (96) can be written as

$$A_{\psi I} = A_{\psi II} \quad \text{at } \eta = \eta_1 \tag{97a}$$

$$A_{\psi II} = A_{\psi III} \quad \text{at } \eta = \eta_2 \tag{97b}$$

$$- \left(\frac{1}{\mu_1}\right) \frac{1}{a\left(\xi_2^2-\nu^2\right)^{\frac{1}{2}}} \frac{\partial}{\partial \xi} \left[\left(\xi^2-1\right)^{\frac{1}{2}} A_{\psi III}\right] \Bigg|_{\xi=\xi_2}$$

$$+ \left(\frac{1}{\mu_1}\right) \frac{1}{a\left(\xi_2^2-\nu^2\right)^{\frac{1}{2}}} \frac{\partial}{\partial \xi} \left[\left(\xi^2-1\right)^{\frac{1}{2}} A_{\psi II}\right] \Bigg|_{\xi=\xi_2} = \sum_{p=1}^{\infty} J_p(\theta) = \sum_{p=1} \frac{KG_p P_p^{1}(\nu)}{a\left(\xi_2^2-\nu^2\right)^{\frac{1}{2}}} \tag{97c}$$

$$\left(\frac{1}{\mu_1}\right)\frac{1}{a\left(\xi_1^2-\nu^2\right)^{\frac{1}{2}}}\frac{\partial}{\partial\xi}\left[\left(\xi^2-1\right)^{\frac{1}{2}}A_{\psi II}\right]\Bigg|_{\xi=\xi_1}=\left(\frac{1}{\mu_2}\right)\frac{1}{a\left(\xi_1^2-\nu^2\right)^{\frac{1}{2}}}\frac{\partial}{\partial\xi}\left[\left(\xi^2-1\right)^{\frac{1}{2}}A_{\psi I}\right]\Bigg|_{\xi=\xi_1} \quad (97d)$$

The general expressions for the potentials in each region (Equations (95)) are then substituted into the boundary conditions (Equations (97)) and are solved for the four constants. Because there are four equations with four unknowns, the potential in each region can be determined. The four boundary value equations are presented below. The index p in the summation sign has <u>both</u> even and odd values and takes on values from 1 to ∞. It is noted at this point that the current density $J_\psi(\theta)$ must be expanded into a set of associated Legendre functions to evaluate the vector potential. The detailed expansion is presented in Reference 8.

$$A_pP_p^1\left(\xi_1\right)P_p^1(\nu)=\left[B_pP_p^1\left(\xi_1\right)+D_pQ_p^1\left(\xi_1\right)\right]P_p^1(\nu) \quad (98a)$$

$$\left[B_pP_p^1\left(\xi_2\right)+D_pQ_p^1\left(\xi_2\right)\right]P_p^1(\nu)=\left[E_pQ_p^1\left(\xi_2\right)\right]P_p^1(\nu) \quad (98b)$$

$$-\left(\frac{1}{\mu_1}\right)\frac{1}{a\left(\xi_2^2-\nu^2\right)^{\frac{1}{2}}}\frac{\partial}{\partial\xi}\left[\left(\xi^2-1\right)^{\frac{1}{2}}\left[E_pQ_p^1(\xi)\right]P_p^1(\nu)\right]\Bigg|_{\xi=\xi_2}$$

$$+\left(\frac{1}{\mu_1}\right)\frac{1}{a\left(\xi_2^2-\nu^2\right)^{\frac{1}{2}}}\frac{\partial}{\partial\xi}\left[\left(\xi^2-1\right)^{\frac{1}{2}}\left[B_pP_p^1(\xi)+D_pQ_p^1(\xi)\right]P_p^1(\nu)\right]\Bigg|_{\xi=\xi_2}=J_p(\theta) \quad (98c)$$

$$\left(\frac{1}{\mu_1}\right)\frac{1}{a\left(\xi_1^2-\nu^2\right)^{\frac{1}{2}}}\frac{\partial}{\partial\xi}\left[\left(\xi^2-1\right)^{\frac{1}{2}}\left[B_pP_p^1(\xi)+D_pQ_p^1(\xi)\right]P_p^1(\nu)\right]\Bigg|_{\xi=\xi_1}$$

$$=\left(\frac{1}{\mu_2}\right)\frac{1}{a\left(\xi_1^2-\nu^2\right)^{\frac{1}{2}}}\frac{\partial}{\partial\xi}\left[\left(\xi^2-1\right)^{\frac{1}{2}}\left[A_pP_p^1(\xi)\right]P_p^1(\nu)\right]\Bigg|_{\xi=\xi_1} \tag{98d}$$

If we make the following substitutions

$$P_p^{\Delta}(\xi)=\frac{\partial}{\partial\xi}\left[\left(\xi^2-1\right)^{\frac{1}{2}}P_p^1(\xi)\right] \tag{99a}$$

$$Q_p^{\Delta}(\xi)=\frac{\partial}{\partial\xi}\left[\left(\xi^2-1\right)^{\frac{1}{2}}Q_p^1(\xi)\right] \tag{99b}$$

and perform simple algebraic manipulation, the four boundary conditions can be simplified to:

$$A_pP_p^1\left(\xi_1\right)=B_pP_p^1\left(\xi_1\right)+D_pQ_p^1\left(\xi_1\right) \tag{100a}$$

$$B_pP_p^1\left(\xi_2\right)+D_pQ_p^1\left(\xi_2\right)=-E_pQ_p^1\left(\xi_2\right) \tag{100b}$$

$$-\left(\frac{1}{\mu_1}\right)\left[E_p Q_p^{\Delta}(\xi_2)\right] + \left(\frac{1}{\mu_1}\right)\left[B_p P_p^{\Delta}(\xi_2) + D_p Q_p^{\Delta}(\xi_2)\right] = \frac{J_p(\theta)\, a\left(\xi_2^2 - \nu^2\right)^{\frac{1}{2}}}{P_p^1(\nu)} \tag{100c}$$

$$\left(\frac{1}{\mu_1}\right)\left[B_p P_p^{\Delta}(\xi_1) + D_p Q_p^{\Delta}(\xi_1)\right] = \left[\frac{1}{\mu_2}\right]\left[A_p P_p^{\Delta}(\xi_1)\right] \tag{100d}$$

The solution of these four simultaneous equations to obtain constants in terms of known quantities gives

$$A_p = \frac{\dfrac{\mu_2}{\mu_1}\left(\dfrac{\mu_1 J_p(\theta)\, a\left(\xi_2^2-\nu^2\right)^{\frac{1}{2}}}{P_p^{\Delta}(\xi_2)\, P_p^1(\nu)}\right)\left(\dfrac{Q_p^1(\xi_2)}{P_p^1(\xi_2)}\right)\left[\dfrac{Q_p^{\Delta}(\xi_1)}{P_p^{\Delta}(\xi_1)} - \dfrac{Q_p^1(\xi_1)}{P_p^1(\xi_1)}\right]}{\left(\dfrac{Q_p^1(\xi_2)}{P_p^1(\xi_2)} - \dfrac{Q_p^{\Delta}(\xi_2)}{P_p^{\Delta}(\xi_2)}\right)\left(\dfrac{\mu_2}{\mu_1}\ \dfrac{Q_p^{\Delta}(\xi_1)}{P_p^{\Delta}(\xi_1)} - \dfrac{Q_p^1(\xi_1)}{P_p^1(\xi_1)}\right)} \tag{101a}$$

$$B_p = \frac{\dfrac{\mu_1 J_p(\theta)\, a\left(\xi_2^2-\nu^2\right)^{\frac{1}{2}}}{P_p^{\Delta}(\xi_2)\, P_p^1(\nu)}\left(\dfrac{Q_p^1(\xi_2)}{P_p^1(\xi_2)}\right)}{\left[\dfrac{Q_p^1(\xi_2)}{P_p^1(\xi_2)} - \dfrac{Q_p^{\Delta}(\xi_2)}{P_p^{\Delta}(\xi_2)}\right]} \tag{101b}$$

$$E_p = \frac{\left(1 - \frac{\mu_2}{\mu_1}\right) \frac{\mu_1 J_p(\theta) a \left(\xi_2^2 - \nu^2\right)^{\frac{1}{2}}}{P_p^{\Delta}(\xi_2) \quad P_p^1(\nu)} \left(\frac{Q_p^1(\xi_2)}{P_p^1(\xi_2)}\right)}{\left[\frac{Q_p^1(\xi_2)}{P_p^1(\xi_2)} - \frac{Q_p^{\Delta}(\xi_2)}{P_p^{\Delta}(\xi_2)}\right]\left[\frac{\mu_2}{\mu_1} \frac{Q_p^{\Delta}(\xi_1)}{P_p^{\Delta}(\xi_1)} - \frac{Q_p^1(\xi_1)}{P_p^1(\xi_1)}\right]}$$

$$+ \frac{\frac{\mu_1 J_p(\theta) a \left(\xi_2^2 - \nu^2\right)^{\frac{1}{2}}}{P_p^{\Delta}(\xi_2) \quad P_p^1(\nu)}}{\left[\frac{Q_p^1(\xi_2)}{P_p^1(\xi_2)} - \frac{Q_p^{\Delta}(\xi_2)}{P_p^{\Delta}(\xi_2)}\right]} \tag{101c}$$

$$D_p = \frac{B_p \left(\frac{\mu_2}{\mu_1} - 1\right)}{\left(\frac{Q_p^1(\xi_1)}{P_p^1(\xi_1)} - \frac{\mu_2}{\mu_1} \frac{Q_p^{\Delta}(\xi_1)}{P_p^{\Delta}(\xi_1)}\right)} \tag{101d}$$

We note that when μ_2 is allowed, in the limit, to approach μ_1, this solution reduces to that of a current band in vacuum. We also note that the solutions are not in the identical form of those given in Reference 10.

The magnetic induction $\overline{B}$ can be determined from

$$B_\theta = \left(\overline{\nabla} \times \overline{A}_\psi\right)_\theta$$

$$B_\eta = \left(\overline{\nabla} \times \overline{A}_\psi\right)_\eta$$

which gives:

$$B_{\eta I} = -\sum_{p=1}^{\infty} \frac{1}{a\left(\xi^2-\nu^2\right)^{\frac{1}{2}}} \frac{\partial}{\partial \nu}\left[\left(1-\nu^2\right)^{\frac{1}{2}} \left(A_p P_p^1(\xi_1)\right) P_p^1(\nu)\right] \tag{102a}$$

$$B_{\theta I} = -\sum_{p=1}^{\infty} \frac{1}{a\left(\xi^2-\nu^2\right)^{\frac{1}{2}}} \frac{\partial}{\partial \xi}\left[\left(\xi^2-1\right)^{\frac{1}{2}} \left(A_p P_p^1(\xi_1)\right) P_p^1(\nu)\right] \tag{102b}$$

$$B_{\eta II} = -\sum_{p=1}^{\infty} \frac{1}{a\left(\xi^2-\nu^2\right)^{\frac{1}{2}}} \frac{\partial}{\partial \nu}\left[\left(1-\nu^2\right)^{\frac{1}{2}} \left(B_p P_p^1(\xi) + D_p Q_p^1(\xi)\right) P_p^1(\nu)\right] \tag{102c}$$

$$B_{\theta II} = -\sum_{p=1}^{\infty} \frac{1}{a\left(\xi^2-\nu^2\right)^{\frac{1}{2}}} \frac{\partial}{\partial \xi}\left[\left(\xi^2-1\right)^{\frac{1}{2}} \left(B_p P_p^1(\xi) + D_p Q_p^1(\xi)\right) P_p^1(\nu)\right] \tag{102d}$$

$$B_{\eta III} = -\sum_{p=1}^{\infty} \frac{1}{a\left(\xi^2-\nu^2\right)^{\frac{1}{2}}} \frac{\partial}{\partial \nu}\left[\left(1-\nu^2\right)^{\frac{1}{2}} \left(E_p Q_p^1(\xi)\right) P_p^1(\nu)\right] \tag{102e}$$

$$B_{\theta III} = - \sum_{p=1}^{\infty} \frac{1}{a\left(\xi^2-\nu^2\right)^{\frac{1}{2}}} \frac{\partial}{\partial \xi} \left[\left(\xi^2-1\right)^{\frac{1}{2}} \left(E_p Q_p^1(\xi)\right) P_p^1(\nu) \right] \qquad (102f)$$

PROLATE SPHEROIDAL SHELL SURROUNDED BY AN INFINITESIMALLY THIN SPHEROIDAL CURRENT BAND

We now proceed to solve the boundary value problem of a ferromagnetic prolate spheroidal shell of homogeneous permeability μ_2 surrounded by an infinitesimally thin prolate spheroidal current band of constant current density $\overline{J}$. The geometry of the problem suggests that a prolate spheroidal coordinate system, as shown in Figure 2, should be used in the solution. Figure 11, a cross section of the problem geometry, identifies the four regions of interest. The boundaries of the prolate spheroidal shell are determined by $\eta = \eta_1$ = constant and $\eta = \eta_2$ = constant. The constant current lies in the boundary $\eta = \eta_3$ = constant. Regions I, III, and IV have a permeability labelled μ_1. Ampere's law states that

$$\overline{\nabla} \times \overline{H} = \overline{J} \qquad (103)$$

and, because $\overline{\nabla} \cdot \overline{B} = 0$, the induction $\overline{B}$ must be the curl of some vector field $\overline{A}$. The governing differential equation for $\overline{A}$, when homogeneous and linear materials are considered, is, from Equation (17),

$$✡\, \overline{A} = - \mu\overline{J} \qquad (104)$$

The general expression in prolate spheroidal coordinates for a current density is

$$\overline{J} = J_\eta \widehat{e}_\eta + J_\theta \widehat{e}_\theta + J_\psi \widehat{e}_\psi \qquad (105)$$

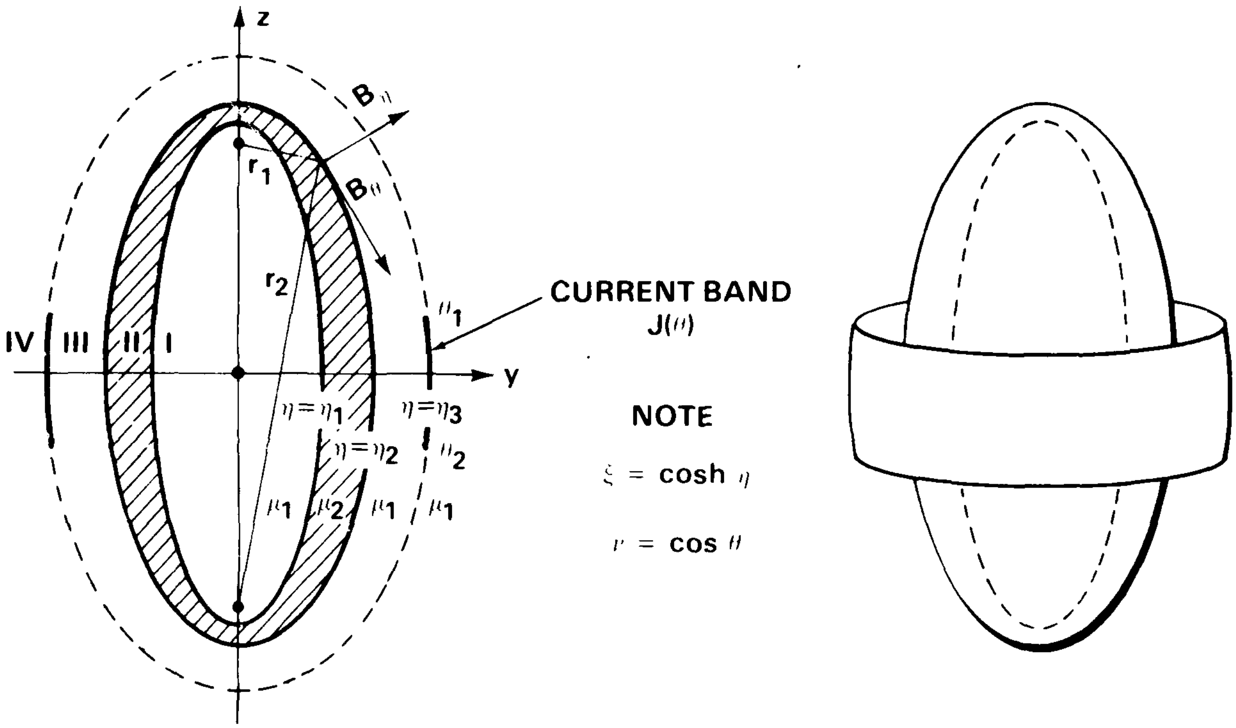


Figure 11 - Ferromagnetic Spheroidal Shell Surrounded by an Infinitesimally Thin Current Band

In the problem presented herein, the current density has only a psi (ψ) component $\left[J_\psi(\theta)\ \hat{e}_\psi\right]$, which means that the vector potential has only a psi component $A_\psi \hat{e}_\psi$. The vector potential $\overline{A} = A_\psi \hat{e}_\psi$ is a function of the prolate spheroidal coordinates η, and θ, i.e., $\left[A_\psi = A_\psi(\eta,\theta)\right]$. The constant current density, which lies on the boundary between regions III and IV, can be expressed by the function

$$\overline{J} = \begin{cases} 0 & \text{, if } \theta < \theta_1 \text{ or } \theta > \theta_2 \\ J_\psi(\theta)\hat{e}_\psi & \text{, if } \theta_1 \leq \theta \leq \theta_2 \end{cases} \tag{106}$$

where $J_\psi(\theta)$ is equal to a constant J along $\eta = \eta_3$ for $\theta_1 \leq \theta \leq \theta_2$.

Therefore, Equation (104) has only an azimuthal component and can be expressed as

$$\nabla^2 \overline{A}_\psi = \nabla^2 \overline{A}_\psi(\eta,\theta) = 0 \quad \begin{pmatrix}\text{in regions I} \\ \text{through IV}\end{pmatrix} \tag{107}$$

When the psi component of the vector Laplacian $\nabla^2 \overline{A}_\psi$ is expanded in prolate spheroidal coordinates, Equation (107) can be expressed as (see Appendix A)

$$\frac{\partial}{\partial\eta}\left[\frac{1}{\sinh\eta}\,\frac{\partial\left(\sinh\eta\ A_\psi\right)}{\partial\eta}\right] + \frac{\partial}{\partial\theta}\left[\frac{1}{\sin\theta}\,\frac{\partial\left(\sin\theta\ A_\psi\right)}{\partial\theta}\right] = 0 \tag{108}$$

(in regions I through IV)

Applying the method of separation of variables, let us assume that A_ψ can be expressed as the product of two functions

$$A_\psi = H(\cosh \eta)\ G(\cos \theta) \tag{109}$$

where H(cosh η) is a function of cosh η only and G(cos θ) is a function of cos θ only. Substituting this form of the component of the vector potential $\bar{A}$ into Equation (108), we have, after separation of variables,

$$\frac{d^2H}{d\eta^2} + \coth \eta \ \frac{dH}{d\eta} - \left(p(p+1) + \frac{1}{\sinh^2 \eta}\right) H = 0 \tag{110a}$$

$$\frac{d^2G}{d\theta^2} + \cot \theta \ \frac{dG}{d\theta} + \left(p(p+1) - \frac{1}{\sin^2 \theta}\right) G = 0 \tag{110b}$$

where the separation constant is p(p+1) and p is an integer from one to infinity. It is well known that differential equations of the form

$$\frac{d^2H'}{d\eta^2} + \coth \eta \ \frac{dH'}{d\eta} - \left(p(p+1) + \frac{m^2}{\sinh^2 \eta}\right) H' = 0 \tag{111a}$$

have the general solution of the form

$$H' = C_1 P_p^m(\cosh \eta) + C_2 Q_p^m(\cosh \eta) \tag{111b}$$

where C_1 and C_2 are constants. It is known that a differential equation of the form

$$\frac{d^2G'}{d\theta^2} + \cot\theta \quad \frac{dG'}{d\theta} + \left[p(p+1) - \frac{m^2}{\sin^2\theta}\right]G' = 0 \tag{112a}$$

has the general solution of the form

$$G' = C_3 P_p^m(\cos\theta) + C_4 Q_p^m(\cos\theta) \tag{112b}$$

where C_3 and C_4 are constants. The associated Legendre functions P_p^m and Q_p^m are of the first and second kind, respectively. Comparison of Equations (110), (111), and (112) shows that in Equations (111) and (112), $m^2=1$. This requires that m always equals unity. The solutions of Equations (110a) and (110b) are expressed as

$$H(\cosh\eta) = A\ P_p^1(\cosh\eta) + B\ Q_p^1(\cosh\eta) \tag{113a}$$

$$G(\cos\theta) = A'\ P_p^1(\cos\theta) + B'\ Q_p^1(\cos\theta) \tag{113b}$$

The general solution of Equation (108) may be formed from the product of solutions to Equations (113a) and (113b) which yield

$$A_\psi = H(\cosh\eta)\ G(\cos\theta) = \sum_{p=1}^{\infty} H_p(\cosh\eta)\ G_p(\cos\theta) \tag{114}$$

$$A_\psi = \sum_{p=1}^{\infty} \left[A\, P_p^1(\cosh\eta) + B\, Q_p^1(\cosh\eta)\right]$$

$$\times \left[A'\; P_p^1(\cos\theta) + B'\; Q_p^1(\cos\theta)\right] \tag{115}$$

For the prolate spheroidal system, the associated Legendre functions of the second kind are infinite at $\cos\theta = \pm 1$ and, as such, cannot be included in a general solution for a given region which includes $\theta = 0$ or $\theta = \pi$. Therefore, in our case, the constant B' is set equal to zero. Equation (115) reduces to

$$A_\psi = \sum_{p=1}^{\infty} \left[K_1 P_p^1(\cosh\eta) + K_2 Q_p^1(\cosh\eta)\right] P_p^1(\cos\theta) \tag{116}$$

where K_1 and K_2 are constants ($K_1 = AA'$, $K_2 = AB'$). When the substitutions $\xi = \cosh\eta$ and $\nu = \cos\theta$ are made in Equation (116), A_ψ can be expressed as

$$A_\psi = \sum_{p=1}^{\infty} \left[K_1 P_p^1(\xi) + K_2 Q_p^1(\xi)\right] P_p^1(\nu) \tag{117}$$

This is the general form of the psi component of the vector potential that will be used to determine the potentials, A_ψ, in each region.

The form of the component of the vector potential A_ψ in regions I through IV is determined from Equation (117). These magnetostatic vector potentials in regions I through IV are:

$$A_{\psi I} = \sum_{p=1}^{\infty} \left[A_p P_p^1(\xi)\right] P_p^1(\nu)$$

$$A_{\psi II} = \sum_{p=1}^{\infty} \left[B_p P_p^1(\xi) + C_p Q_p^1(\xi)\right] P_p^1(\nu)$$

$$A_{\psi III} = \sum_{p=1}^{\infty} \left[D_p P_p^1(\xi) + E_p Q_p^1(\xi)\right] P_p^1(\nu) \tag{118}$$

$$A_{\psi IV} = \sum_{p=1}^{\infty} \left[F_p Q_p^1(\xi)\right] P_p^1(\nu)$$

Because the potential must be finite in each of the regions I, II, and III and approach zero as $\xi \to \infty$ in region IV, the following constants were set equal to zero.

1. For $A_{\psi I}$, the constant associated with $Q_p^1(\xi)\ P_p^1(\nu)$ was set equal to zero because

$$Q_p^1(\xi) \to \infty \text{ at } \xi = +1$$

2. For $A_{\psi IV}$ the constant associated with $P_p^1(\xi)\ P_p^1(\nu)$ was set equal to zero because $P_p^1(\xi) \to \infty$ as $\xi \to \infty$.

$$(\text{We note } Q_p^1(\xi) \to 0 \text{ as } \xi \to \infty)$$

The constants A_p, B_p, C_p, D_p, E_p, and F_p are to be determined from the boundary conditions. At each interface, the basic laws of magnetostatics (Equations (3a)) reduce to boundary conditions on $\overline{B}$ and $\overline{H}$ that can be used to evaluate these six constants. The normal component of $\overline{B}$ across each boundary must be continuous, i.e., $(\overline{B}_2-\overline{B}_1) \cdot \overline{n}_{12} = 0$ where the quantity $\overline{n}_{12}$ is the unit outward normal to the surface. This provides the following boundary conditions which must be satisfied by the solutions given in Equation (118) for each region.

$$B_{\eta I} = B_{\eta II} \quad \text{at } \eta = \eta_1 \tag{119a}$$

$$B_{\eta II} = B_{\eta III} \quad \text{at } \eta = \eta_2 \tag{119b}$$

$$B_{\eta III} = B_{\eta IV} \quad \text{at } \eta = \eta_3 \tag{119c}$$

The eta (η) or normal component of the magnetic field (B_η) is expressed in terms of the vector potential as

$$B_\eta = \left(\overline{\nabla} x \overline{A}_\psi\right)_\eta = \frac{1}{e_2 e_3} \frac{\partial\left(e_3 A_\psi\right)}{\partial\theta}$$

(120)

$$= - \frac{1}{a\left(\xi^2-\nu^2\right)^{\frac{1}{2}}} \frac{\partial}{\partial\nu}\left[\left(1-\nu^2\right)^{\frac{1}{2}} A_\psi\right]$$

where $\overline{B} = \overline{\nabla} \times \overline{A}$

$$= \frac{1}{a\left(\sinh^2 \eta + \sin^2 \theta\right)(\sinh \eta \sin \theta)} \quad x$$

(Note: Above equation continued on next page).

$$
\begin{vmatrix}
\hat{e}_\eta \left(\sinh^2 \eta + \sin^2 \theta\right)^{\frac{1}{2}} & \hat{e}_\theta \left(\sinh^2 \eta + \sin^2 \theta\right)^{\frac{1}{2}} & \hat{e}_\psi \sinh \eta \sin \theta \\
\frac{\partial}{\partial \eta} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \psi} \\
0 & 0 & A_\psi \sinh \eta \sin \theta
\end{vmatrix}
$$

and

$$\xi = \cosh \eta, \; e_1 = e_2 = a\left(\sinh^2 \eta + \sin^2 \theta\right)^{\frac{1}{2}} = a\left(\xi^2 - \nu^2\right)^{\frac{1}{2}}$$

$$\nu = \cos \theta, \; e_3 = a \sinh \eta \sin \theta$$

However, because the vector potentials in each region are functions of $P_p^1(\nu)$, we can simplify Equation (119) to constraints on A_ψ at the interfaces:

$$A_{\psi I} = A_{\psi II} \quad \text{at } \eta = \eta_1 \tag{121a}$$

$$A_{\psi II} = A_{\psi III} \quad \text{at } \eta = \eta_2 \tag{121b}$$

$$A_{\psi III} = A_{\psi IV} \quad \text{at } \eta = \eta_3 \tag{121c}$$

The second set of boundary conditions st..tes that the theta (θ) or tangential component of $\overline{H}$ across each boundary must satisfy the relationship

$$\overline{n}_{12} \times \left(\overline{H}_2 - \overline{H}_1\right) = \overline{J}_s \tag{122}$$

where $\overline{J}_s$ (which equals $J_\psi(\theta)\hat{e}_\psi$) is the real surface current density in the limit of vanishing width between the two regions. Using the relationship $\overline{B} = \mu\overline{H}$, Equation (122) can be expressed as

$$\frac{B_{\theta 2}}{\mu_2} - \frac{B_{\theta 1}}{\mu_1} = J_\psi(\theta) \tag{123}$$

Referring to the curl in Equation (120), we can write B_θ in the form

$$B_\theta = (\overline{\nabla}\mathrm{x}\overline{A}_\psi)_\theta = \frac{-1}{e_1 e_3}\,\frac{\partial\left(e_3 A_\psi\right)}{\partial\eta} = -\,\frac{1}{a\left(\xi^2-\nu^2\right)^{\frac{1}{2}}}\,\frac{\partial}{\partial\xi}\left[\left(\xi^2-1\right)^{\frac{1}{2}} A_\psi\right] \tag{124}$$

From Equations (123) and (124) the tangential components of $\overline{B}$ in regions I through IV must satisfy the relationships:

$$\left(\frac{1}{\mu_2}\right)\frac{1}{a\left(\xi_1^2-\nu^2\right)^{\frac{1}{2}}}\,\frac{\partial}{\partial\xi}\left[\left(\xi^2-1\right)^{\frac{1}{2}} A_{\psi II}\right]\Bigg|_{\xi=\xi_1}$$

$$= \left(\frac{1}{\mu_1}\right)\frac{1}{a\left(\xi_1^2-\nu^2\right)^{\frac{1}{2}}}\,\frac{\partial}{\partial\xi}\left[\left(\xi^2-1\right)^{\frac{1}{2}} A_{\psi I}\right]\Bigg|_{\xi=\xi_1} \tag{125a}$$

$$\left(\frac{1}{\mu_1}\right)\frac{1}{a\left(\xi_2^2-\nu^2\right)^{\frac{1}{2}}}\frac{\partial}{\partial\xi}\left[\left(\xi^2-1\right)^{\frac{1}{2}}A_{\psi III}\right]\Bigg|_{\xi=\xi_2}$$

$$=\left(\frac{1}{\mu_2}\right)\frac{1}{a\left(\xi_2^2-\nu^2\right)^{\frac{1}{2}}}\frac{\partial}{\partial\xi}\left[\left(\xi^2-1\right)^{\frac{1}{2}}A_{\psi II}\right]\Bigg|_{\xi=\xi_2} \tag{125b}$$

$$-\left(\frac{1}{\mu_1}\right)\frac{1}{a\left(\xi_3^2-\nu^2\right)^{\frac{1}{2}}}\frac{\partial}{\partial\xi}\left[\left(\xi^2-1\right)^{\frac{1}{2}}A_{\psi IV}\right]\Bigg|_{\xi=\xi_3}$$

$$+\left(\frac{1}{\mu_1}\right)\frac{1}{a\left(\xi_3^2-\nu^2\right)^{\frac{1}{2}}}\frac{\partial}{\partial\xi}\left[\left(\xi^2-1\right)^{\frac{1}{2}}A_{\psi III}\right]\Bigg|_{\xi=\xi_3}$$

$$=\sum_{=1}^{\infty}J_p(\theta)=\sum_{p=1}^{\infty}\frac{KG_pP_p^1(\nu)}{a\left(\xi_3^2-\nu^2\right)^{\frac{1}{2}}} \tag{125c}$$

The general expressions for the potentials in each region (Equation (118)) are then substituted into the boundary conditions (Equations (121) and (125)) and solved for the six constants (A_p, B_p, C_p, D_p, E_p, and F_p). Because there are six equations with six unknowns, the potential in each region can be determined. The six boundary value equations are presented below. The index p in the summation sign has both even and odd values and takes on values from 1 to ∞. It is noted at this point that the current density $J_\psi(\theta)$ must be expanded into a set of associated Legendre functions to evaluate the constants in the vector potential (Equation (118)). The detailed expansion is presented in Appendix B of Reference 8. The six expressions for the boundary conditions are:

$$A_p P_p^1 \left(\xi_1\right) P_p^1(\nu) = \left[B_p P_p^1\left(\xi_1\right) + C_p Q_p^1\left(\xi_1\right)\right] P_p^1(\nu) \tag{126a}$$

$$\left(\frac{1}{\mu_2}\right) \frac{1}{a\left(\xi_1^2-\nu^2\right)^{\frac{1}{2}}} \frac{\partial}{\partial \xi} \left[\left(\xi^2-1\right)^{\frac{1}{2}}\left(B_p P_p^1(\xi) + C_p Q_p^1(\xi)\right) P_p^1(\nu)\right] \Bigg|_{\xi=\xi_1}$$

$$= \left(\frac{1}{\mu_1}\right) \frac{1}{a\left(\xi_1^2-\nu^2\right)^{\frac{1}{2}}} \frac{\partial}{\partial \xi} \left[\left(\xi^2-1\right)^{\frac{1}{2}}\left(A_p P_p^1(\xi)\right) P_p^1(\nu)\right] \Bigg|_{\xi=\xi_1} \tag{126b}$$

$$\left[B_p P_p^1\left(\xi_2\right) + C_p Q_p^1\left(\xi_2\right)\right] P_p^1(\nu) = \left[D_p P_p^1\left(\xi_2\right) + E_p Q_p^1\left(\xi_2\right)\right] P_p^1(\nu) \tag{126c}$$

$$\left(\frac{1}{\mu_1}\right) \frac{1}{a\left(\xi_2^2-\nu^2\right)^{\frac{1}{2}}} \frac{\partial}{\partial \xi} \left[\left(\xi^2-1\right)^{\frac{1}{2}}\left(D_p P_p^1(\xi) + E_p Q_p^1(\xi)\right) P_p^1(\nu)\right] \Bigg|_{\xi=\xi_2}$$

$$= \left(\frac{1}{\mu_2}\right) \frac{1}{a\left(\xi_2^2-\nu^2\right)^{\frac{1}{2}}} \frac{\partial}{\partial \xi} \left[\left(\xi^2-1\right)^{\frac{1}{2}}\left(B_p P_p^1(\xi) + C_p Q_p^1(\xi)\right) P_p^1(\nu)\right] \Bigg|_{\xi=\xi_2} \tag{126d}$$

$$\left[D_p P_p^1(\xi_3) + E_p Q_p^1(\xi_3)\right] P_p^1(\nu) = F_p Q_p^1(\xi_3) P_p^1(\nu) \tag{126e}$$

$$- \left(\frac{1}{\mu_1}\right) \frac{1}{a\left(\xi_3^2-\nu^2\right)^{\frac{1}{2}}} \frac{\partial}{\partial\xi} \left[\left(\xi^2-1\right)^{\frac{1}{2}}\left(F_p Q_p^1(\xi)\right) P_p^1(\nu)\right] \Bigg|_{\xi=\xi_3}$$

$$+ \left(\frac{1}{\mu_1}\right) \frac{1}{a\left(\xi_3^2-\nu^2\right)^{\frac{1}{2}}} \frac{\partial}{\partial\xi} \left[\left(\xi^2-1\right)^{\frac{1}{2}}\left(D_p P_p^1(\xi) + E_p Q_p^1(\xi)\right) P_p^1(\nu)\right] \Bigg|_{\xi=\xi_3}$$

$$= J_p(\theta) = \frac{KG_p P_p^1(\nu)}{a\left(\xi_3^2-\nu^2\right)^{\frac{1}{2}}} \tag{126f}$$

If we make the following substitution

$$P_p^{\Delta}(\xi) = \frac{d}{d\xi}\left[\left(\xi^2-1\right)^{\frac{1}{2}} P_p^1(\xi)\right] \tag{127}$$

$$Q_p^{\Delta}(\xi) = \frac{d}{d\xi}\left[\left(\xi^2-1\right)^{\frac{1}{2}} Q_p^1(\xi)\right] \tag{128}$$

and perform simple algebraic manipulations, the six boundary conditions can be simplified to:

$$A_p P_p^1(\xi_1) = B_p P_p^1(\xi_1) + C_p Q_p^1(\xi_1)$$

$$\left(\frac{1}{\mu_2}\right)\left[B_p P_p^{\Delta}(\xi_1) + C_p Q_p^{\Delta}(\xi_1)\right] = \left(\frac{1}{\mu_1}\right) A_p P_p^{\Delta}(\xi_1) \tag{129b}$$

$$B_p P_p^1(\xi_2) + C_p Q_p^1(\xi_2) = D_p P_p^1(\xi_2) + E_p Q_p^1(\xi_2) \tag{129c}$$

$$\left(\frac{1}{\mu_1}\right)\left[D_p P_p^{\Delta}(\xi_2) + E_p Q_p^{\Delta}(\xi_2)\right] = \left(\frac{1}{\mu_2}\right)\left[B_p P_p^{\Delta}(\xi_2) + C_p Q_p^{\Delta}(\xi_2)\right] \tag{129d}$$

$$D_p P_p^1(\xi_3) + E_p Q_p^1(\xi_3) = F_p Q_p^1(\xi_3) \tag{129e}$$

$$-\left(\frac{1}{\mu_1}\right) F_p Q_p^{\Delta}(\xi_3) + \left(\frac{1}{\mu_1}\right) D_p P_p^{\Delta}(\xi_3) + \left(\frac{1}{\mu_1}\right) E_p Q_p^{\Delta}(\xi_3) = \frac{J_p(\theta)\, a\left(\xi_3^2-\nu^2\right)^{\frac{1}{2}}}{P_p^1(\nu)} \tag{129f}$$

where
$$J_p(\theta) = \frac{K\; G_p P_p^1(\nu)}{a\left(\xi_3^2-\nu^2\right)^{\frac{1}{2}}}$$

The solution of these six simultaneous equations to obtain E_p in terms of known quantities gives:

$$E_p = \frac{-\frac{1}{\mu_2}\left([x]J_p^{II}P_p^{\Delta}(\xi_2)\right) - \frac{1}{\mu_2}\left(J_p^{II}\right)Q_p^{\Delta}(\xi_2) + \frac{1}{\mu_1}\left(J_p^{I}\right)P_p^{\Delta}(\xi_2)}{\frac{1}{\mu_2}\left([x][z]P_p^{\Delta}(\xi_2)\right) + \frac{1}{\mu_2}\left([z]Q_p^{\Delta}(\xi_2)\right) - \left(\frac{1}{\mu_1}\right)Q_p^{\Delta}(\xi_2)} \qquad (130a)$$

where

$$[x] = \frac{\left(\left(\frac{\mu_1}{\mu_2}\right)\frac{Q_p^{\Delta}(\xi_1)}{P_p^{\Delta}(\xi_1)} - \frac{Q_p^{1}(\xi_1)}{P_p^{1}(\xi_1)}\right)}{\left(1 - \frac{\mu_1}{\mu_2}\right)} \qquad (130b)$$

$$[z] = \frac{Q_p^{1}(\xi_2)}{[x]P_p^{1}(\xi_2) + Q_p^{1}(\xi_2)} \qquad (130c)$$

$$J_p(\theta) = \frac{K_p G_p P_p^{1}(\nu)}{a\left(\xi_3^2-\nu^2\right)^{\frac{1}{2}}}, \qquad \text{for } (\nu = \cos\theta) \qquad (130d)$$

$$J_p^{I} = \frac{\left\{\frac{-\mu_1 J_p(\theta)\, a\left(\xi_3^2-\nu^2\right)^{\frac{1}{2}}}{P_p^{1}(\nu)Q_p^{\Delta}(\xi_3)}\right\}}{\left(\frac{P_p^{1}(\xi_3)}{Q_p^{1}(\xi_3)} - \frac{P_p^{\Delta}(\xi_3)}{Q_p^{\Delta}(\xi_3)}\right)} \qquad (130e)$$

$$J_p^{II} = \frac{J_p^I P_p^1(\xi_2)}{[x]P_p^1(\xi_2) + Q_p^1(\xi_2)} \tag{130f}$$

The numerical values for the other five coefficients can be obtained from the following equations:

$$C_p = J_p^{II} + E_p[z] \tag{131a}$$

$$B_p = [x]C_p \tag{131b}$$

$$D_p = J_p^I \tag{131c}$$

$$A_p = B_p + C_p \frac{Q_p^1(\xi_1)}{P_p^1(\xi_1)} \tag{131d}$$

$$F_p = \frac{D_p P_p^1(\xi_3)}{Q_p^1(\xi_3)} + E_p \tag{131e}$$

Because the six coefficients can be determined for a specified problem from Equations (130) and (131), the potentials $A_{\psi I}$, $A_{\psi II}$, $A_{\psi III}$, and $A_{\psi IV}$, in regions I through IV can be completely determined. The normal (B_η) and tangential (B_θ) to the surface η = constant (or ξ = constant) components of the magnetic induction in each region I through IV can be determined by using Equations (120) and (124), to be:

$$B_{\theta I} = -\sum_{p=1}^{\infty} \frac{1}{a\left(\xi^2-\nu^2\right)^{\frac{1}{2}}} \frac{\partial}{\partial \xi}\left[\left(\xi^2-1\right)^{\frac{1}{2}} A_p P_p^1(\xi) P_p^1(\nu)\right] \tag{132a}$$

$$B_{\eta I} = -\sum_{p=1}^{\infty} \frac{1}{a\left(\xi^2-\nu^2\right)^{\frac{1}{2}}} \frac{\partial}{\partial \nu}\left[\left(1-\nu^2\right)^{\frac{1}{2}} A_p P_p^1(\xi) P_p^1(\nu)\right] \tag{132b}$$

$$B_{\theta II} = -\sum_{p=1}^{\infty} \frac{1}{a\left(\xi^2-\nu^2\right)^{\frac{1}{2}}} \frac{\partial}{\partial \xi}\left[\left(\xi^2-1\right)^{\frac{1}{2}}\left(B_p P_p^1(\xi) + C_p Q_p^1(\xi)\right) P_p^1(\nu)\right] \tag{132c}$$

$$B_{\eta II} = -\sum_{p=1}^{\infty} \frac{1}{a\left(\xi^2-\nu^2\right)^{\frac{1}{2}}} \frac{\partial}{\partial \nu}\left[\left(1-\nu^2\right)^{\frac{1}{2}}\left(B_p P_p^1(\xi) + C_p Q_p^1(\xi)\right) P_p^1(\nu)\right] \tag{132d}$$

$$B_{\theta III} = -\sum_{p=1}^{\infty} \frac{1}{a\left(\xi^2-\nu^2\right)^{\frac{1}{2}}} \frac{\partial}{\partial \xi}\left[\left(\xi^2-1\right)^{\frac{1}{2}}\left(D_p P_p^1(\xi) + E_p Q_p^1(\xi)\right) P_p^1(\nu)\right] \tag{132e}$$

$$B_{\eta III} = -\sum_{p=1}^{\infty} \frac{1}{a\left(\xi^2-\nu^2\right)^{\frac{1}{2}}} \frac{\partial}{\partial \nu}\left[\left(1-\nu^2\right)^{\frac{1}{2}}\left(D_p P_p^1(\xi) + E_p Q_p^1(\xi)\right) P_p^1(\nu)\right] \tag{132f}$$

$$B_{\theta IV} = - \sum_{p=1}^{\infty} \frac{1}{a\left(\xi^2-\nu^2\right)^{\frac{1}{2}}} \frac{\partial}{\partial \xi} \left[\left(\xi^2-1\right)^{\frac{1}{2}}\left(F_p Q_p^1(\xi) P_p^1(\nu)\right)\right] \tag{132g}$$

$$B_{\eta IV} = - \sum_{p=1}^{\infty} \frac{1}{a\left(\xi^2-\nu^2\right)^{\frac{1}{2}}} \frac{\partial}{\partial \nu} \left[\left(1-\nu^2\right)^{\frac{1}{2}} \left(F_p Q_p^1(\xi)\right) P_p^1(\nu)\right] \tag{132h}$$

PROLATE SPHEROIDAL SHELL SURROUNDING AN INFINITESIMALLY THIN SPHEROIDAL CURRENT BAND

We now proceed to solve the boundary value problem of a ferromagnetic spheroidal shell of homogeneous permeability μ_2, surrounding an infinitesimally thin prolate spheroidal current band having a constant current density $\overline{J}$. Figure 12 shows the cross section of the problem geometry. The coordinate system shown previously in Figure 2 will be used in the solution. The boundaries of the prolate spheroidal shell are determined by $\eta = \eta_3$ and $\eta = \eta_2$. The steady state current lies in the boundary $\eta = \eta_1$ and between $\theta_1 \le \theta \le \theta_2$. As in the previous problem, the constant current density has only a psi component $J_\psi(\theta)\widehat{e}_\psi$, and thus the vector potential has onl· psi component $A_\psi \widehat{e}_\psi$. The vector potential is a function of the prolate spheroı . coordinates η and θ. The constant current density is expressed by Equation (106) when the boundary η is changed to $\eta = \eta_1$.

The governing partial differential equation has only a psi component and is given by

$$\text{✡}\,\overline{A} = \text{✡}\,\overline{A}_\psi(\eta,\theta) = 0 \quad \text{(in regions I through IV)} \tag{133}$$

When the vector Laplacian ✡ $\overline{A}$ is expanded in prolate spheroidal coordinates, Equation (133) can be expressed as (see Appendix A of Reference 8)

$$\frac{\partial}{\partial \eta}\left[\frac{1}{\sinh \eta} \frac{\partial(\sinh \eta \, A_\psi)}{\partial \eta}\right] + \frac{\partial}{\partial \theta}\left[\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (\sin \theta \, A_\psi)\right] = 0 \tag{134}$$

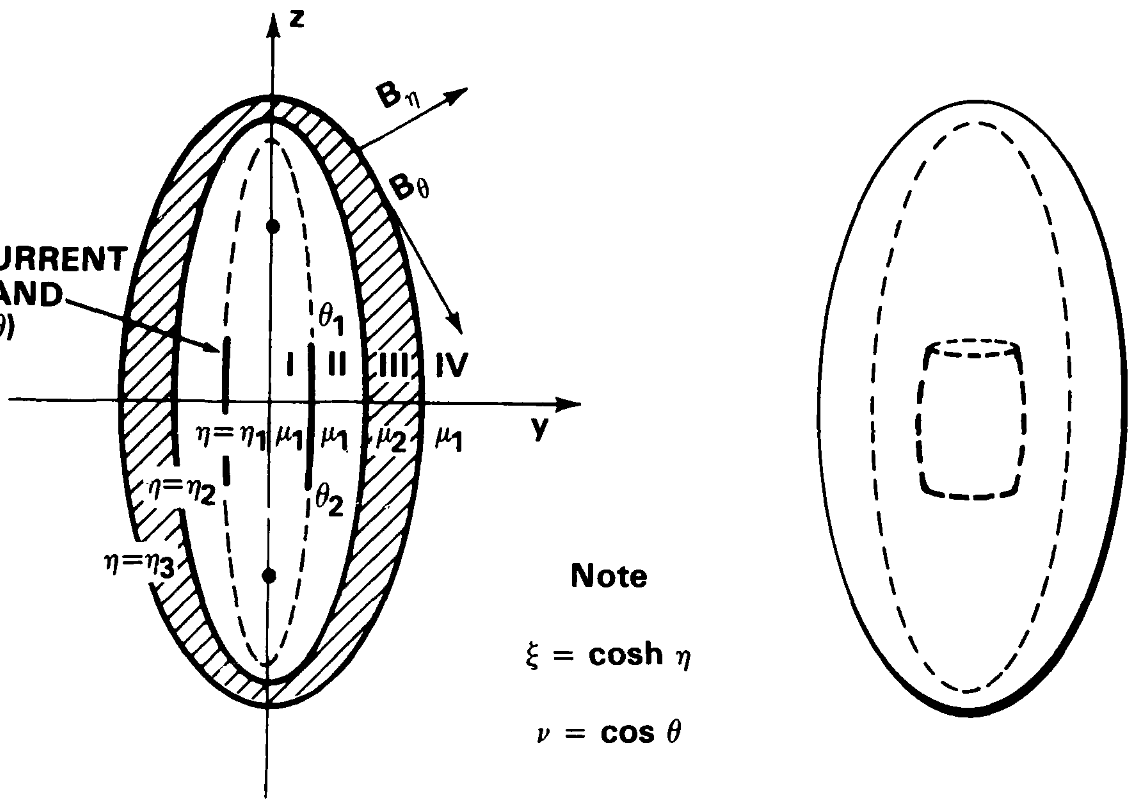


Figure 12 - Infinitesimally Thin Current Band Surrounded by a Ferromagnetic Spheroidal Shell

Adopting the following notation

$$\xi = \cosh \eta, \ \nu = \cos \theta \tag{135}$$

and following the logic presented earlier, the solutions for A_ψ in regions I through IV have the general form

$$A_\psi = \sum_{p=1}^{\infty} \left[K_1 P_p^1(\xi) + K_2 Q_p^1(\xi) \right] P_p^1(\nu) \tag{136}$$

The form of the components of the vector potential A_ψ in each of the regions I through IV is determined from Equation (136). These components of the vector potential in each region are:

$$A_{\psi I} = \sum_{p=1}^{\infty} \left[H_p P_p^1(\xi) \right] P_p^1(\nu) \tag{137a}$$

$$A_{\psi II} = \sum_{p=1}^{\infty} \left[I_p P_p^1(\xi) + K_p Q_p^1(\xi) \right] P_p^1(\nu) \tag{137b}$$

$$A_{\psi III} = \sum_{p=1}^{\infty} \left[L_p P_p^1(\xi) + M_p Q_p^1(\xi) \right] P_p^1(\nu) \tag{137c}$$

$$A_{\psi IV} = \sum_{p=1}^{\infty} \left[N_p Q_p^1(\xi) \right] P_p^1(\nu) \tag{137d}$$

The P_p^1 functions are the associated Legendre functions of the first kind of degree 1 and order p, and the Q_p^1 functions are associated Legendre functions of the second kind.

At each interface, the basic laws of magnetostatics reduce to boundary conditions on $\overline{B}$ and $\overline{H}$ (see Equations (119) and (122)) that can be used to determine related boundary conditions on $\overline{A}$:

$$A_{\psi I} = A_{\psi II} \qquad \eta = \eta_1 \tag{138a}$$

$$A_{\psi II} = A_{\psi III} \qquad \eta = \eta_2 \tag{138b}$$

$$A_{\psi III} = A_{\psi IV} \qquad \eta = \eta_3 \tag{138c}$$

$$-\left(\frac{1}{\mu_1}\right)\frac{1}{a\left(\xi_1^2-\nu^2\right)^{\frac{1}{2}}}\frac{\partial}{\partial\xi}\left[\left(\xi^2-1\right)^{\frac{1}{2}}A_{\psi II}\right]\Bigg|_{\xi=\xi_1}$$

$$+\left(\frac{1}{\mu_1}\right)\frac{1}{a\left(\xi_1^2-\nu^2\right)^{\frac{1}{2}}}\frac{\partial}{\partial\xi}\left[\left(\xi^2-1\right)^{\frac{1}{2}}A_{\psi I}\right]\Bigg|_{\xi=\xi_1} = \sum_{p=1}^{\infty} J_p(\theta) \tag{138d}$$

$$\left(\frac{1}{\mu_1}\right)\frac{1}{a\left(\xi_2^2-\nu^2\right)^{\frac{1}{2}}}\frac{\partial}{\partial\xi}\left[\left(\xi^2-1\right)^{\frac{1}{2}}A_{\psi II}\right]\Bigg|_{\xi=\xi_2}$$

$$=\left(\frac{1}{\mu_2}\right)\frac{1}{a\left(\xi_2^2-\nu^2\right)^{\frac{1}{2}}}\frac{\partial}{\partial\xi}\left[\left(\xi^2-1\right)^{\frac{1}{2}}A_{\psi III}\right]\Bigg|_{\xi=\xi_2} \qquad (138e)$$

$$\left(\frac{1}{\mu_2}\right)\frac{1}{a\left(\xi_3^2-\nu^2\right)^{\frac{1}{2}}}\frac{\partial}{\partial\xi}\left[\left(\xi^2-1\right)^{\frac{1}{2}}A_{\psi III}\right]\Bigg|_{\xi=\xi_3}$$

$$=\left(\frac{1}{\mu_1}\right)\frac{1}{a\left(\xi_3^2-\nu^2\right)^{\frac{1}{2}}}\frac{\partial}{\partial\xi}\left[\left(\xi^2-1\right)^{\frac{1}{2}}A_{\psi IV}\right]\Bigg|_{\xi=\xi 3} \qquad (138f)$$

These boundary conditions are then used to evaluate the constants in Equation (137). Using Equations (137) and (138) to solve for the coefficients (where the index p takes on all values from 1 to ∞) we get:

$$H_p P_p^1(\xi_1) P_p^1(\nu) = \left[I_p P_p^1(\xi_1) + K_p Q_p^1(\xi_1)\right] P_p^1(\nu) \qquad (139a)$$

$$-\left(\frac{1}{\mu_1}\right)\frac{1}{a\left(\xi_1^2-\nu^2\right)^{\frac{1}{2}}}\ \frac{\partial}{\partial\xi}\left[\left(\xi^2-1\right)^{\frac{1}{2}}\left(I_pP_p^1(\xi)+K_pQ_p(\xi)\right)P_p^1(\nu)\right]\Bigg|_{\xi=\xi_1}$$

$$+\left(\frac{1}{\mu_1}\right)\frac{1}{a\left(\xi_1^2-\nu^2\right)^{\frac{1}{2}}}\ \frac{\partial}{\partial\xi}\left[\left(\xi^2-1\right)^{\frac{1}{2}}\left(H_pP_p^1(\xi)P_p^1(\nu)\right)\right]\Bigg|_{\xi=\xi_1}=J_p(\theta)=\frac{K\ G_pP_p^1(\nu)}{a\left(\xi_1^2-\nu^2\right)^{\frac{1}{2}}} \tag{139b}$$

$$\left[I_pP_p^1(\xi_2)+K_pQ_p^1(\xi_2)\right]P_p^1(\nu)=\left[L_pP_p^1(\xi_2)+M_pQ_p^1(\xi_2)\right]P_p^1(\nu) \tag{139c}$$

$$\left(\frac{1}{\mu_2}\right)\frac{1}{a\left(\xi_2^2-\nu^2\right)^{\frac{1}{2}}}\ \frac{\partial}{\partial\xi}\left[\left(\xi^2-1\right)^{\frac{1}{2}}\left(L_pP_p^1(\xi)+M_pQ_p^1(\xi)\right)P_p^1(\nu)\right]\Bigg|_{\xi=\xi_2}$$

$$=\left(\frac{1}{\mu_1}\right)\frac{1}{a\left(\xi_2^2-\nu^2\right)^{\frac{1}{2}}}\ \frac{\partial}{\partial\xi}\left[\left(\xi^2-1\right)^{\frac{1}{2}}\left(I_pP_p^1(\xi)+K_pQ_p^1(\xi)\right)P_p^1(\nu)\right]\Bigg|_{\xi=\xi_2} \tag{139d}$$

$$\left[L_pP_p^1(\xi_3)+M_pQ_p^1(\xi_3)\right]P_p^1(\nu)=N_pQ_p^1(\xi_3)P_p^1(\nu) \tag{139e}$$

$$\left(\frac{1}{\mu_1}\right) \frac{1}{a\left(\xi_3^2-\nu^2\right)^{\frac{1}{2}}} \frac{\partial}{\partial \xi}\left[\left(\xi^2-1\right)^{\frac{1}{2}}\left(N_p Q_p^1(\xi)\right) P_p^1(\nu)\right]\Bigg|_{\xi=\xi_3}$$

$$= \left(\frac{1}{\mu_2}\right) \frac{1}{a\left(\xi_3^2-\nu^2\right)^{\frac{1}{2}}} \frac{\partial}{\partial \xi}\left[\left(\xi^2-1\right)^{\frac{1}{2}}\left(L_p P_p^1(\xi) + M_p Q_p^1(\xi)\right) P_p^1(\nu)\right]\Bigg|_{\xi=\xi_3} \qquad (139f)$$

If we make the following substitutions

$$P_p^{\Delta}(\xi) = \frac{\partial}{\partial \xi}\left[\left(\xi^2-1\right)^{\frac{1}{2}} P_p^1(\xi)\right] \qquad (140)$$

$$Q_p^{\Delta}(\xi) = \frac{\partial}{\partial \xi}\left[\left(\xi^2-1\right)^{\frac{1}{2}} Q_p^1(\xi)\right] \qquad (141)$$

and perform simple algebraic manipulations, the six boundary conditions reduce to:

$$H_p P_p^1(\xi_1) = I_p P_p^1(\xi_1) + K_p Q_p^1(\xi_1) \qquad (142a)$$

$$-\left(\frac{1}{\mu_1}\right)\left(I_p P_p^{\Delta}(\xi_1) + K_p Q_p^{\Delta}(\xi_1)\right) + \left(\frac{1}{\mu_1}\right) H_p P_p^{\Delta}(\xi_1) = \frac{J_p(\theta)\, a\left(\xi_1^2-\nu^2\right)^{\frac{1}{2}}}{P_p^1(\nu)} \qquad (142b)$$

$$I_p P_p^1(\xi_2) + K_p Q_p^1(\xi_2) = L_p P_p^1(\xi_2) + M_p Q_p^1(\xi_2) \tag{142c}$$

$$\left(\frac{1}{\mu_2}\right)\left(L_p P_p^{\Delta}(\xi_2) + M_p Q_p^{\Delta}(\xi_2)\right) = \left(\frac{1}{\mu_1}\right)\left(I_p P_p^{\Delta}(\xi_2) + K_p Q_p^{\Delta}(\xi_2)\right) \tag{142d}$$

$$L_p P_p^1(\xi_3) + M_p Q_p^1(\xi_3) = N_p Q_p^1(\xi_3) \tag{142e}$$

$$\left(\frac{1}{\mu_1}\right) N_p Q_p^{\Delta}(\xi_3) = \left(\frac{1}{\mu_2}\right)\left(L_p P_p^{\Delta}(\xi_3) + M_p Q_p^{\Delta}(\xi_3)\right) \tag{142f}$$

It should be noted in the above equations that the current density $J_\psi(\theta)$ was expanded into a set of associated Legendre functions to evalute the constants in the vector potential components (see Appendix B of Reference 8).

The solution of these six simultaneous Equations (142a) through (142f) to obtain L_p in terms of known quantities is:

$$L_p = \frac{\left(\frac{1}{\mu_1}\right)\left[J_p^{II} P_p^{\Delta}(\xi_2) + J_p^{I} Q_p^{\Delta}(\xi_2)\right]}{\left(-\frac{1}{\mu_1}\right)[V] P_p^{\Delta}(\xi_2) + \left(\frac{1}{\mu_2}\right) P_p^{\Delta}(\xi_2) + \left(\frac{1}{\mu_2}\right)[U] Q_p^{\Delta}(\xi_2)} \tag{143}$$

where
$$[U] = \frac{\left[\frac{P_p^1(\xi_3)}{Q_p^1(\xi_3)} - \frac{P_p^\Delta(\xi_3)}{Q_p^\Delta(\xi_3)}\left(\frac{\mu_1}{\mu_2}\right)\right]}{\left[\frac{\mu_1}{\mu_2} - 1\right]} \tag{144}$$

$$[V] = \frac{P_p^1(\xi_2) + [U] Q_p^1(\xi_2)}{P_p^1(\xi_2)} \tag{145}$$

$$J_p(\theta) = \frac{K\ G_p P_p^1(\nu)}{a\left(\xi_1^2 - \nu^2\right)^{\frac{1}{2}}} \tag{146}$$

$$J_p^I = \frac{\dfrac{\mu_1 J_p(\theta)\, a\left(\xi_1^2 - \nu^2\right)^{\frac{1}{2}}}{P_p^\Delta(\xi_1) P_p^1(\nu)}}{\left[\frac{Q_p^1(\xi_1)}{P_p^1(\xi_1)} - \frac{Q_p^\Delta(\xi_1)}{P_p^\Delta(\xi_1)}\right]} \tag{147}$$

$$J_p^{II} = -J_p^I \frac{Q_p^1(\xi_2)}{P_p^1(\xi_2)} \tag{148}$$

The numerical values for the other five coefficients can be obtained from the following equations:

$$K_p = J_p^I \tag{149}$$

$$M_p = L_p[U] \tag{150}$$

$$I_p = J_p^{II} = L_p[V] \tag{151}$$

$$N_p = L_p \frac{P_p^1(\xi_3)}{Q_p^1(\xi_3)} + M_p \tag{152}$$

$$H_p = I_p + \frac{K_p Q_p^1(\xi_1)}{P_p^1(\xi_1)} \tag{153}$$

The components of the potential A_ψ in regions I through IV can be determined because the coefficients H_p, I_p, K_p, L_p, M_p, and N_p can be calculated for a specific problem. The normal (B_η) and tangential (B_θ) components (to the surface η = constant or ξ = constant) of the magnetic induction in each region (I through IV) can be determined by using Equations (120) and (124), to be:

$$B_{\theta I} = - \sum_{p=1}^{\infty} \frac{1}{a\left(\xi^2-\nu^2\right)^{1/2}} \frac{\partial}{\partial \xi}\left[\left(\xi^2-1\right)^{1/2}\left(H_p P_p^1(\xi)\right) P_p^1(\nu)\right] \tag{154a}$$

$$B_{\eta I} = -\sum_{p=1}^{\infty} \frac{1}{a\left(\xi^2-\nu^2\right)^{\frac{1}{2}}} \frac{\partial}{\partial\nu}\left[\left(1-\nu^2\right)^{\frac{1}{2}}\left(H_p P_p^1(\xi)\right) P_p^1(\nu)\right] \tag{154b}$$

$$B_{\theta II} = -\sum_{p=1}^{\infty} \frac{1}{a\left(\xi^2-\nu^2\right)^{\frac{1}{2}}} \frac{\partial}{\partial\xi}\left[\left(\xi^2-1\right)^{\frac{1}{2}}\right.$$

$$\left. \times \left(I_p P_p^1(\xi) + K_p Q_p^1(\xi)\right) P_p^1(\nu)\right] \tag{154c}$$

$$B_{\eta II} = -\sum_{p=1}^{\infty} \frac{1}{a\left(\xi^2-\nu^2\right)^{\frac{1}{2}}} \frac{\partial}{\partial\nu}\left[\left(1-\nu^2\right)^{\frac{1}{2}}\right.$$

$$\left. \times \left(I_p P_p^1(\xi) + K_p Q_p^1(\xi)\right) P_p^1(\nu)\right] \tag{154d}$$

$$B_{\theta III} = -\sum_{p=1}^{\infty} \frac{1}{a\left(\xi^2-\nu^2\right)^{\frac{1}{2}}} \frac{\partial}{\partial\xi}\left[\left(\xi^2-1\right)^{\frac{1}{2}}\right.$$

$$\left. \times \left(L_p P_p^1(\xi) + M_p Q_p^1(\xi)\right) P_p^1(\nu)\right] \tag{154e}$$

$$B_{\eta III} = - \sum_{p=1}^{\infty} \frac{1}{a\left(\xi^2-\nu^2\right)^{\frac{1}{2}}} \frac{\partial}{\partial\nu}\left[\left(1-\nu^2\right)^{\frac{1}{2}}\right.$$

$$\left. \times \left(L_p P_p^1(\xi) + M_p Q_p^1(\xi)\right) P_p^1(\nu)\right] \tag{154f}$$

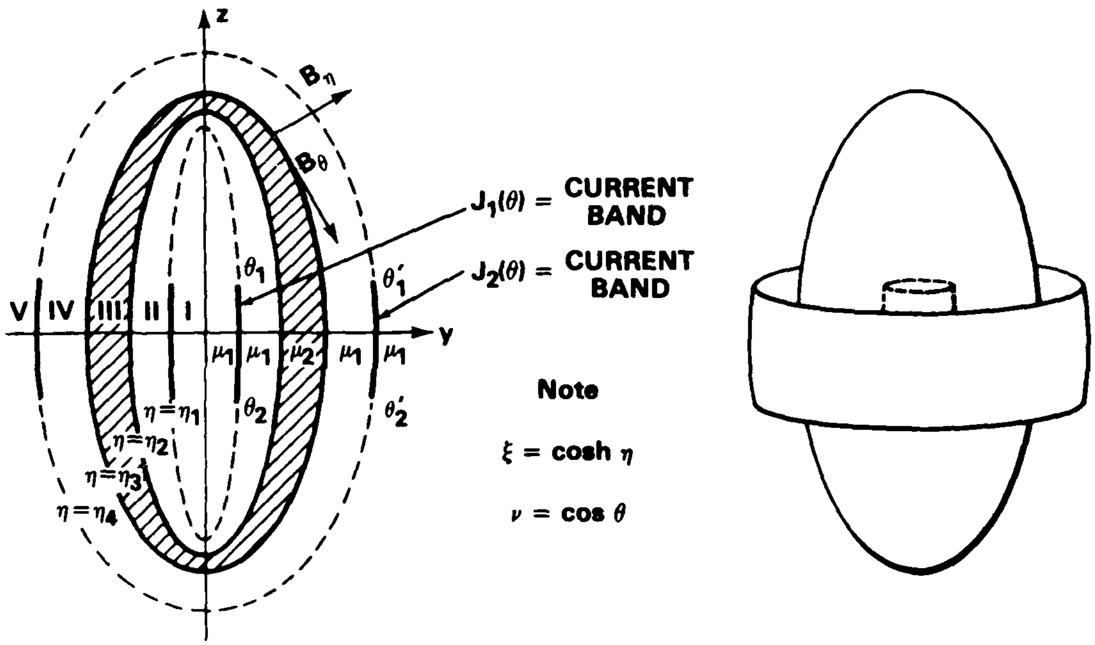
$$B_{\theta IV} = - \sum_{p=1}^{\infty} \frac{1}{a\left(\xi^2-\nu^2\right)^{\frac{1}{2}}} \frac{\partial}{\partial\xi}\left[\left(\xi^2-1\right)^{\frac{1}{2}}\right.$$

$$\left. \times \left(N_p Q_p^1(\xi)\right) P_p^1(\nu)\right] \tag{154g}$$

$$B_{\eta IV} = - \sum_{p=1}^{\infty} \frac{1}{a\left(\xi^2-\nu^2\right)^{\frac{1}{2}}} \frac{\partial}{\partial\nu}\left[\left(1-\nu^2\right)^{\frac{1}{2}} \left(N_p Q_p^1(\xi)\right) P_p^1(\nu)\right] \tag{154h}$$

PROLATE SPHEROIDAL SHELL WITH INTERNAL AND EXTERNAL INFINITESIMALLY THIN SPHEROIDAL CURRENT BANDS

We now proceed to solve the boundary value problem of a ferromagnetic prolate spheroidal shell of homogeneous permeability μ_2 with internal and external, infinitesimally thin, prolate spheroidal current bands of constant current density $\overline{J}_1$ and $\overline{J}_2$, respectively. The geometry of the problem suggests that a prolate spheroidal coordinate system as shown in Figure 2 can be used in the problem solution. Figure 13, a cross section of the problem geometry, identifies the five regions of interest. The boundaries of the prolate spheroidal shell are determined by $\eta = \eta_2$ and $\eta = \eta_3$, constants. The direct currents lie in the boundaries $\eta = \eta_4$, and $\eta = \eta_1$, constants. Regions I, II, IV, and V have a permeability equal to vacuum μ_o, which, for convenience, will be labelled μ_1. In the problem presented here, the current densities have only a psi (ψ) component $\left[J_\psi(\theta)\hat{e}_\psi\right]$, which means that the vector potential has only a psi component $A_\psi \hat{e}_\psi$. The vector potential $\left[A = A_\psi \hat{e}_\psi\right]$ is a function of the prolate spheroidal coordinates $\eta,\theta\left[\text{i.e., } A_\psi = A_\psi\ (\eta,\theta)\right]$. The constant current densities, which lie on the boundaries between regions I and II and between regions IV and V, can be expressed by the functions



Page 99 - Figure 13 - Ferromagnetic Spheroidal Shell Surrounding and Surrounded by Infinitesimally Thin Current Bands

$$\overline{J}_1 = \begin{cases} 0, & \text{if } \theta < \theta_1 \text{ or } \theta > \theta_2 \\ J_{\psi 1}(\theta)\,\hat{e}_\psi, & \text{if } \theta_1 \le \theta \le \theta_2 \end{cases} \tag{155}$$

where $J_{\psi 1}(\theta)$ is equal to a constant J_1 along $\eta = \eta_1$ for $\theta_1 \le \theta \le \theta_2$ and

$$\overline{J}_2 = \begin{cases} 0, & \text{if } \theta < \theta_1' \text{ or } \theta > \theta_2' \\ J_{\psi 2}(\theta)\,\hat{e}_\psi, & \text{if } \theta_1' \le \theta \le \theta_2' \end{cases} \tag{156}$$

where $J_{\psi 2}(\theta)$ is equal to a constant J_2 along $\eta = \eta_4$ for $\theta_1' \le \theta \le \theta_2'$. Therefore, Equation (17) has only an azimuthal or psi component and can be expressed as

$$\nabla^2 \overline{A} = \nabla^2 \overline{A}_\psi(\eta,\theta) = 0 \text{ in regions I through V} \tag{157}$$

When the vector Laplacian $\nabla^2 \overline{A}_\psi$ is expanded in prolate spheroidal coordinates, Equation (157) can be expressed (see Appendix A, Reference 8)

$$\frac{\partial}{\partial \eta}\left[\frac{1}{\sinh \eta}\,\frac{\partial\,(\sinh \eta\, A_\psi)}{\partial \eta}\right] + \frac{\partial}{\partial \theta}\left[\frac{1}{\sin \theta}\,\frac{\partial\,(\sin \theta\, A_\psi)}{\partial \theta}\right] = 0 \tag{158}$$

Adopting the following notation

$$\xi = \cosh \eta,\ \nu = \cos \theta$$

and following the logic presented earlier, the solutions for A_ψ in regions I through IV have the general form

$$A_\psi = \sum_{p=1}^{\infty} \left[K_1 P_p^1(\xi) + K_2 Q_p^1(\xi) \right] P_p^1(\nu) \tag{159}$$

The form of the component of the vector potential A_ψ in regions I through V is determined from Equation (159). These magnetostatic vector potentials in regions I through V are

$$A_{\psi I} = \sum_{p=1}^{\infty} \left[A_p P_p^1(\xi) \right] P_p^1(\nu) \tag{160a}$$

$$A_{\psi II} = \sum_{p=1}^{\infty} \left[B_p P_p^1(\xi) + C_p Q_p^1(\xi) \right] P_p^1(\nu) \tag{160b}$$

$$A_{\psi III} = \sum_{p=1}^{\infty} \left[D_p P_p^1(\xi) + E_p Q_p^1(\xi) \right] P_p^1(\nu) \tag{160c}$$

$$A_{\psi IV} = \sum_{p=1}^{\infty} \left[F_p P_p^1(\xi) + G_p Q_p^1(\xi) \right] P_p^1(\nu) \tag{160d}$$

$$A_{\psi V} = \sum_{p=1}^{\infty} \left[H_p Q_p^1(\xi) \right] P_p^1(\nu) \tag{160e}$$

Because the potential must be finite in each of regions I through IV, and approach zero as $\xi \to \infty$ in region V, the following constants were set equal to zero:

1. For $A_{\psi I}$, the constant associated with $Q_p^1(\xi)\ P_p^1(\nu)$ was set equal to zero because

$$Q_p^1(\xi) \to \infty \text{ at } \xi = 1 \text{ (z axis between } \pm a)$$

2. For $A_{\psi V}$, the constant associated with $P_p^1(\xi)$ $P_p^1(\nu)$ was set equal to zero because

$$P_p^1(\xi) \to \infty \text{ as } \xi \to \infty$$

(we note $Q_p^1(\xi) \to 0$ as $\xi \to \infty$).

Constants A_p, B_p, C_p, D_p, E_p, F_p, G_p, and H_p are to be determined from the boundary conditions. At each interface, the basic laws of magnetostatics (Equations (3a)) reduce to boundary conditions on $\overline{B}$ and $\overline{H}$ that can be used to evaluate these eight constants. The normal component of $\overline{B}$ across each boundary must be continuous, i.e., $(\overline{B}_2-\overline{B}_1) \cdot \overline{n}_{12} = 0$ where the quantity $\overline{n}_{12}$ is the unit outward normal to the surface. This provides the following boundary conditions which must be satisfied by the solutions given in Equation (160) for each region

$$B_{\eta I} = B_{\eta II} \quad \text{at} \quad \eta = \eta_1 \tag{161a}$$

$$B_{\eta II} = B_{\eta III} \quad \text{at} \quad \eta = \eta_2 \tag{161b}$$

$$B_{\eta III} = B_{\eta IV} \quad \text{at} \quad \eta = \eta_3 \tag{161c}$$

$$B_{\eta IV} = B_{\eta V} \quad \text{at} \quad \eta = \eta_4 \tag{161d}$$

However, because the vector potentials in each region are functions of $P_p^1(\nu)$, we can simplify Equation (161) to constraints on A_ψ at the interfaces

$$A_{\psi I} = A_{\psi II} \quad \text{at} \quad \eta = \eta_1 \tag{162a}$$

$$A_{\psi II} = A_{\psi III} \quad \text{at} \quad \eta = \eta_2 \tag{162b}$$

$$A_{\psi III} = A_{\psi IV} \quad \text{at} \quad \eta = \eta_3 \tag{162c}$$

$$A_{\psi IV} = A_{\psi V} \quad \text{at} \quad \eta = \eta_4 \tag{162d}$$

The second set of boundary conditions states that the theta (θ) or tangential, component of $\overline{H}$ across each boundary must satisfy the relationship

$$\overline{n}_{12} \times (\overline{H}_2 - \overline{H}_1) = \overline{J}_s \tag{163}$$

where $\overline{J}_s$ $\left(\text{which equals } J_\psi(\theta)\right)$ is the real surface current density in the limit of vanishing width between the two regions. Using the relationship $\overline{B} = \mu\overline{H}$, Equation (163) can be expressed as

$$\frac{B_{\theta 2}}{\mu_2} - \frac{B_{\theta 1}}{\mu_1} = J_\psi(\theta) \tag{164}$$

The current must be expanded in a series of associated Legendre functions $P_p^1(\nu)$ as in Reference 9. The form of the current is

$$J_\psi(\theta) = \frac{J \sum_{p=1}^{\infty} V_p P_p^1 (\cos\theta)}{a(\sinh^2\eta + \sin^2\theta)^{\frac{1}{2}}} \tag{165}$$

where, using $\xi = \cosh\eta$ and $\nu = \cos\theta$, V_p can be shown to be

$$V_p = \frac{-(2p+1)\ a}{2p(p+1)} \int_{\nu_1}^{\nu_2} (\xi^2 - \nu^2)^{\frac{1}{2}}\ P_p^1(\nu) d\nu \tag{166}$$

For the two current bands of interest, we have

$$J_{\psi 1}(\theta) = \frac{J_1 \sum_{p=1}^{\infty} V_p P_p^1(\nu)}{a(\xi_1^2 - \nu^2)^{\frac{1}{2}}} \tag{167a}$$

where $$V_p = \frac{-(2p+1)\ a}{2p(p+1)} \int_{\nu_1}^{\nu_2} \left(\xi_1^2-\nu^2\right)^{\frac{1}{2}} P_p^1(\nu)d\nu \tag{167b}$$

and $$J_{\psi 2}(\theta) = \frac{J_2 \sum_{p=1}^{\infty} U_p P_p^1(\nu)}{a\left(\xi_4^2-\nu^2\right)^{\frac{1}{2}}} \tag{168a}$$

where $$U_p = \frac{-(2p+1)\ a}{2p(p+1)} \int_{\nu_1'}^{\nu_2'} \left(\xi_4^2-\nu^2\right)^{\frac{1}{2}} P_p^1(\nu)\ d\nu \tag{168b}$$

and $\nu_1' = \cos\theta_1'$ and $\nu_2' = \cos\theta_2'$.

Referring to the curl in Equation (120), we can write B_θ in the form

$$B_\theta = (\overline{\nabla}\mathrm{x}\overline{A}_\psi)_\theta = -\frac{1}{e_1 e_3}\frac{\partial(e_3 A_\psi)}{\partial\eta} = -\frac{1}{a\left(\xi^2-\nu^2\right)^{\frac{1}{2}}}\frac{\partial}{\partial\xi}\left[\left(\xi^2-1\right)^{\frac{1}{2}} A_\psi\right] \tag{169}$$

From Equations (164) and (169), the tangential components of $\overline{B}$ in regions I through V must satisfy the relationship

$$+\left(\frac{1}{\mu_1}\right)\frac{1}{a(\xi_1^2-\nu^2)^{\frac{1}{2}}}\ \frac{\partial}{\partial\xi}\left[(\xi^2-1)^{\frac{1}{2}}\ A_{\psi I}\right]\Bigg|_{\xi=\xi_1}$$

$$-\left(\frac{1}{\mu_1}\right)\frac{1}{a(\xi_1^2-\nu^2)^{\frac{1}{2}}}\ \frac{\partial}{\partial\xi}\left[(\xi^2-1)^{\frac{1}{2}}\ A_{\psi II}\right]\Bigg|_{\xi=\xi_1}$$

$$= J_{p1}(\theta) = \frac{J_1 \sum_{p=1}^{\infty} V_p P_p^1(\nu)}{a(\xi_1^2-\nu^2)^{\frac{1}{2}}} \tag{170a}$$

$$\left(\frac{1}{\mu_2}\right)\frac{1}{a(\xi_2^2-\nu^2)^{\frac{1}{2}}}\ \frac{\partial}{\partial\xi}\left[(\xi^2-1)^{\frac{1}{2}}\ A_{\psi III}\right]\Bigg|_{\xi=\xi_2}$$

$$=\left(\frac{1}{\mu_1}\right)\frac{1}{a(\xi_2^2-\nu^2)^{\frac{1}{2}}}\ \frac{\partial}{\partial\xi}\left[(\xi^2-1)^{\frac{1}{2}}\ A_{\psi II}\right]\Bigg|_{\xi=\xi_2} \tag{170b}$$

$$\left(\frac{1}{\mu_1}\right)\frac{1}{a(\xi_3^2-\nu^2)^{\frac{1}{2}}}\ \frac{\partial}{\partial\xi}\left[(\xi^2-1)^{\frac{1}{2}}\ A_{\psi IV}\right]\Bigg|_{\xi=\xi_3}$$

$$=\left(\frac{1}{\mu_2}\right)\frac{1}{a(\xi_3^2-\nu^2)^{\frac{1}{2}}}\ \frac{\partial}{\partial\xi}\left[(\xi^2-1)^{\frac{1}{2}}\ A_{\psi III}\right]\Bigg|_{\xi=\xi_3} \tag{170c}$$

$$+ \left(\frac{1}{\mu_1}\right) \frac{1}{a(\xi_4^2-\nu^2)^{\frac{1}{2}}} \frac{\partial}{\partial \xi} \left[(\xi^2-1)^{\frac{1}{2}} A_{\psi IV} \right] \Bigg|_{\xi=\xi_4}$$

$$-\left(\frac{1}{\mu_1}\right) \frac{1}{a(\xi_4^2-\nu^2)^{\frac{1}{2}}} \frac{\partial}{\partial \xi} \left[(\xi^2-1)^{\frac{1}{2}} A_{\psi V} \right] \Bigg|_{\xi=\xi_4}$$

$$= J_{p2}(\theta) = \frac{J_2 \sum_{p=1}^{\infty} U_p P_p^1(\nu)}{a(\xi_4^2-\nu^2)^{\frac{1}{2}}} \qquad (170d)$$

The general expressions for the potentials in each region (Equation (160)) are then substituted into the boundary conditions (Equations (162) and (170)) and solved for the eight constants (A_p, B_p, C_p, D_p, E_p, F_p, G_p, and H_p). Because there are eight equations with eight unknowns, the potential in each region can be determined. The eight boundary value equations are presented below. The index, p, in the summation sign has both even and odd values. The eight expressions for the boundary conditions are:

$$A_p P_p^1(\xi_1) P_p^1(\nu) = \left[B_p P_p^1(\xi_1) + C_p Q_p^1(\xi_1) \right] P_p^1(\nu) \qquad (171a)$$

$$-\left(\frac{1}{\mu_1}\right)\frac{1}{a(\xi_1^2-\nu^2)^{\frac{1}{2}}}\ \frac{\partial}{\partial\xi}\ \left[(\xi^2-1)^{\frac{1}{2}}\ \left(B_pP_p^1(\xi)\ +\ C_pQ_p^1(\xi)\right)P_p^1(\nu)\right]\Bigg|_{\xi=\xi_1}$$

$$+\left(\frac{1}{\mu_1}\right)\frac{1}{a(\xi_1^2-\nu^2)^{\frac{1}{2}}}\ \frac{\partial}{\partial\xi}\ \left[(\xi^2-1)^{\frac{1}{2}}\ \left(A_pP_p^1(\xi)\right)P_p^1(\nu)\right]\Bigg|_{\xi=\xi_1}\ =\ J_{p1}(\theta) \qquad (171b)$$

$$\left[B_pP_p^1(\xi_2)\ +\ C_pQ_p^1(\xi_2)\right]\ P_p^1(\nu)\ =\left[\ D_pP_p^1(\xi_2)\ +\ E_pQ_p^1(\xi_2)\right]\ P_p^1(\nu) \qquad (171c)$$

$$\left(\frac{1}{\mu_2}\right)\frac{1}{a(\xi_2^2-\nu^2)^{\frac{1}{2}}}\ \frac{\partial}{\partial\xi}\ \left[\ (\xi^2-1)^{\frac{1}{2}}\left(D_pP_p^1(\xi)\ +\ E_pQ_p^1(\xi)\right)\ P_p^1(\nu)\right]\Bigg|_{\xi=\xi_2}$$

$$=\ \left(\frac{1}{\mu_1}\right)\frac{1}{a(\xi_2^2-\nu^2)^{\frac{1}{2}}}\ \frac{\partial}{\partial\xi}\ \left[(\xi^2-1)^{\frac{1}{2}}\left(B_pP_p^1(\xi)\ +\ C_pQ_p^1(\xi)\right)P_p^1(\nu)\right]\Bigg|_{\xi=\xi_2} \qquad (171d)$$

$$\left[D_pP_p^1(\xi_3)\ +\ E_pQ_p^1(\xi_3)\right]\ P_p^1(\nu)\ =\left[\ F_pP_p^1(\xi_3)\ +\ G_pQ_p^1(\xi_3)\right]\ P_p^1(\nu) \qquad (171e)$$

$$\left(\frac{1}{\mu_1}\right)\frac{1}{a(\xi_3^2-\nu^2)^{\frac{1}{2}}}\ \frac{\partial}{\partial\xi}\left[(\xi^2-1)^{\frac{1}{2}}\left(F_pP_p^1(\xi)+G_pQ_p^1(\xi)\right)P_p^1(\nu)\right]\Bigg|_{\xi=\xi_3}$$

$$=\left(\frac{1}{\mu_2}\right)\frac{1}{a(\xi_3^2-\nu^2)^{\frac{1}{2}}}\ \frac{\partial}{\partial\xi}\left[(\xi^2-1)^{\frac{1}{2}}\left(D_pP_p^1(\xi)+E_pQ_p^1(\xi)\right)P_p^1(\nu)\right]\Bigg|_{\xi=\xi_3} \tag{171f}$$

$$\left[F_pP_p^1(\xi_4)+G_pQ_p^1(\xi_4)\right]P_p^1(\nu)=H_pQ_p^1(\xi_4)P_p^1(\nu) \tag{171g}$$

$$\left(-\frac{1}{\mu_1}\right)\frac{1}{a(\xi_4^2-\nu^2)^{\frac{1}{2}}}\ \frac{\partial}{\partial\xi}\left[(\xi^2-1)^{\frac{1}{2}}\left(H_pQ_p^1(\xi)\right)P_p^1(\nu)\right]\Bigg|_{\xi=\xi_4}$$

$$+\left(\frac{1}{\mu_1}\right)\frac{1}{a(\xi_4^2-\nu^2)^{\frac{1}{2}}}\ \frac{\partial}{\partial\xi}\left[(\xi^2-1)^{\frac{1}{2}}\left(F_pP_p^1(\xi)+G_pQ_p^1(\xi)\right)P_p^1(\nu)\right]\Bigg|_{\xi=\xi_4}=J_{p2}(\theta) \tag{171h}$$

By making the following substitutions

$$P_p^{\Delta}(\xi)=\frac{\partial}{\partial\xi}\left[(\xi^2-1)^{\frac{1}{2}}\ P_p^1(\xi)\right] \tag{172a}$$

$$Q_p^{\Delta}(\xi)=\frac{\partial}{\partial\xi}\left[(\xi^2-1)^{\frac{1}{2}}\ Q_p^1(\xi)\right] \tag{172b}$$

and performing algebraic manipulation, the eight boundary conditions can be simplified to

$$A_p P_p^1(\xi_1) = B_p P_p^1(\xi_1) + C_p Q_p^1(\xi_1) \tag{173a}$$

$$-\left(\frac{1}{\mu_1}\right)\left[B_p P_p^{\Delta}(\xi_1) + C_p Q_p^{\Delta}(\xi_1)\right] + \left(\frac{1}{\mu_1}\right)\left[A_p P_p^{\Delta}(\xi_1)\right]$$

$$= \frac{J_{p1}(\theta)\ a(\xi_1^2-\nu^2)^{\frac{1}{2}}}{P_p^1(\nu)} \tag{173b}$$

$$B_p P_p^1(\xi_2) + C_p Q_p^1(\xi_2) = D_p P_p^1(\xi_2) + E_p Q_p^1(\xi_2) \tag{173c}$$

$$\left(\frac{1}{\mu_2}\right)\left[D_p P_p^{\Delta}(\xi_2) + E_p Q_p^{\Delta}(\xi_2)\right] = \left(\frac{1}{\mu_1}\right)\left[B_p P_p^{\Delta}(\xi_2) + C_p Q_p^{\Delta}(\xi_2)\right] \tag{173d}$$

$$D_p P_p^1(\xi_3) + E_p Q_p^1(\xi_3) = F_p P_p^1(\xi_3) + G_p Q_p^1(\xi_3) \tag{173e}$$

$$\left(\frac{1}{\mu_2}\right)\left[D_p P_p^{\Delta}(\xi_3) + E_p Q_p^{\Delta}(\xi_3)\right] = \left(\frac{1}{\mu_1}\right)\left[F_p P_p^{\Delta}(\xi_3) + G_p Q_p^{\Delta}(\xi_3)\right] \tag{173f}$$

$$\left[F_p P_p^1(\xi_4) + G_p Q_p^1(\xi_4)\right] = H_p Q_p^1(\xi_4) \tag{173g}$$

$$\left(-\frac{1}{\mu_1}\right)\left(H_p Q_p^{\Delta}(\xi_4)\right) + \left(\frac{1}{\mu_1}\right)\left(F_p P_p^{\Delta}(\xi_4) + G_p Q_p^{\Delta}(\xi_4)\right)$$

$$= \frac{J_{p2}(\theta)\ a(\xi_4^2-\nu^2)^{\frac{1}{2}}}{P_p^1(\nu)}$$

The solution of these eight simultaneous equations to obtain the constants gives:

$$A_p = B_p + C_p \frac{Q_p^1(\xi_1)}{P_p^1(\xi_1)} \tag{174a}$$

$$C_p = \frac{\left[\dfrac{\mu_1 J_{p1}(\theta)\ a\left(\xi_1^2-\nu^2\right)^{\frac{1}{2}}}{P_p^1(\nu)\ P_p^{\Delta}(\xi_1)}\right]}{\left[\dfrac{Q_p^1(\xi_1)}{P_p^1(\xi_1)} - \dfrac{Q_p^{\Delta}(\xi_1)}{P_p^{\Delta}(\xi_1)}\right]} = J_{p1}^I \tag{174b}$$

$$F_p = E_p \frac{Q_p^1(\xi_3)}{P_p^1(\xi_3)} - G_p \frac{Q_p^1(\xi_3)}{P_p^1(\xi_3)} + D_p \tag{174c}$$

$$E_p = D_p\ [Q] - G_p\ [R] \tag{174d}$$

where
$$[Q] = \frac{\left(\frac{\mu_1}{\mu_2} - 1\right)}{\left[\frac{Q_p^1(\xi_3)}{P_p^1(\xi_3)} - \left(\frac{\mu_1}{\mu_2}\right)\frac{Q_p^{\Delta}(\xi_3)}{P_p^{\Delta}(\xi_3)}\right]}$$

$$[R] = \frac{\left[\frac{Q_p^{\Delta}(\xi_3)}{P_p^{\Delta}(\xi_3)} - \frac{Q_p^1(\xi_3)}{P_p^1(\xi_3)}\right]}{\left[\frac{Q_p^1(\xi_3)}{P_p^1(\xi_3)} - \left(\frac{\mu_1}{\mu_2}\right)\frac{Q_p^{\Delta}(\xi_3)}{P_p^{\Delta}(\xi_3)}\right]}$$

$$B_p = J_{p1}^{II} + D_p[S] - G_p[Z] \tag{174e}$$

where

$$J_{p1}^{II} = J_{p1}^{I}\,\frac{Q_p^1(\xi_2)}{P_p^1(\xi_2)}\,, \quad J_{p1}^{I} = \frac{\left[\frac{\mu_1\, J_{p1}(\theta)\; a(\xi_1^2-\nu^2)^{\frac{1}{2}}}{P_p^1(\nu)\; P_p^{\Delta}(\xi_1)}\right]}{\left[\frac{Q_p^1(\xi_1)}{P_p^1(\xi_1)} - \frac{Q_p^{\Delta}(\xi_1)}{P_p^{\Delta}(\xi_1)}\right]}$$

$$[S] = \left(1 + [Q] \; \frac{Q_p^1(\xi_2)}{P_p^1(\xi_2)}\right)$$

$$[Z] = \left([R] \; \frac{Q_p^1(\xi_2)}{P_p^1(\xi_2)}\right) .$$

$$D_p = J_{p1}^{III} + [U] \; G_p \tag{174f}$$

where
$$J_{p1}^{III} = \frac{\left(\frac{1}{\mu_1}\right) J_{p1}^{I} \; Q_p^{\Delta}(\xi_2) + \left(\frac{1}{\mu_1}\right) J_{p1}^{II} \; P_p^{\Delta}(\xi_2)}{\left[\left(-\frac{1}{\mu_1}\right) [S] \; P_p^{\Delta}(\xi_2) + \left(\frac{1}{\mu_2}\right) P_p^{\Delta}(\xi_2) + \left(\frac{1}{\mu_2}\right) [Q] \; Q_p^{\Delta}(\xi_2)\right]}$$

$$[U] = \frac{\left(-\frac{1}{\mu_1}\right) [Z] \; P_p^{\Delta}(\xi_2) + \left(\frac{1}{\mu_2}\right) [R] \; Q_p^{\Delta}(\xi_2)}{\left[\left(-\frac{1}{\mu_1}\right) [S] \; P_p^{\Delta}(\xi_2) + \left(\frac{1}{\mu_2}\right) P_p^{\Delta}(\xi_2) + \left(\frac{1}{\mu_2}\right) [Q] \; Q_p^{\Delta}(\xi_2)\right]}$$

$$H_p = G_p + J_{p2}^{II} \tag{174g}$$

where
$$J_{p2}^{II} = \frac{J_{p2}^{I}}{\big[[A]-[C]\big]}$$

$$[A] = \frac{Q_p^1(\xi_4)}{P_p^1(\xi_4)}$$

$$[C] = \frac{Q_p^\Delta(\xi_4)}{P_p^\Delta(\xi_4)}$$

$$G_p = \frac{-J_{p1}^{III}\,[Q]\,[H] - J_{p1}^{III} + J_{p2}^{II}\,[A]}{[U]\,[Q]\,[H] - [R]\,[H] - [H] + [U]} \tag{174h}$$

where
$$J_{p1}^{III} = \frac{\left(\frac{1}{\mu_1}\right) J_p^I\, Q_p^\Delta(\xi_2) + \left(\frac{1}{\mu_1}\right) J_{p1}^{II}\, P_p^\Delta(\xi_2)}{\left[\left(-\frac{1}{\mu_1}\right)[S]\, P_p^\Delta(\xi_2) + \left(\frac{1}{\mu_2}\right) P_p^\Delta(\xi_2) + \left(\frac{1}{\mu_2}\right)[Q]\, Q_p^\Delta(\xi_2)\right]}$$

$$J_{p2}^{I} = \frac{\mu_1\, J_{p2}(0)\; a(\xi_4^2-\nu^2)^{1/2}}{P_p^\Delta(\xi_4)\; P_p^1(\nu)}$$

$$[H] = \frac{Q_p^1(\xi_3)}{P_p^1(\xi_3)}$$

Because the eight coefficients can be determined for a specified problem from Equation (174), the potentials A_I, A_{II}, A_{III}, A_{IV}, and A_V in regions I through V can be completely determined. The normal (B_η) and tangential (B_θ) components of the magnetic induction in each region, I through V, can be determined by using Equations (120) and (124) to be:

$$B_{\theta I} = - \sum_{p=1}^{\infty} \frac{1}{a\left(\xi^2-\nu^2\right)^{\frac{1}{2}}} \frac{\partial}{\partial \xi} \left[\left(\xi^2-1\right)^{\frac{1}{2}} \left(A_p P_p^1(\xi)\right) P_p^1(\nu) \right] \tag{175a}$$

$$B_{\eta I} = - \sum_{p=1}^{\infty} \frac{1}{a\left(\xi^2-\nu^2\right)^{\frac{1}{2}}} \frac{\partial}{\partial \nu} \left[\left(1-\nu^2\right)^{\frac{1}{2}} \left(A_p P_p^1(\xi)\right) P_p^1(\nu) \right] \tag{175b}$$

$$B_{\theta II} = - \sum_{p=1}^{\infty} \frac{1}{a\left(\xi^2-\nu^2\right)^{\frac{1}{2}}} \frac{\partial}{\partial \xi} \left[\left(\xi^2-1\right)^{\frac{1}{2}} \left(B_p P_p^1(\xi) + C_p Q_p^1(\xi)\right) P_p^1(\nu) \right] \tag{175c}$$

$$B_{\eta II} = - \sum_{p=1}^{\infty} \frac{1}{a\left(\xi^2-\nu^2\right)^{\frac{1}{2}}} \frac{\partial}{\partial \nu} \left[\left(1-\nu^2\right)^{\frac{1}{2}} \left(B_p P_p^1(\xi) + C_p Q_p^1(\xi)\right) P_p^1(\nu) \right] \tag{175d}$$

$$B_{\theta III} = - \sum_{p=1}^{\infty} \frac{1}{a\left(\xi^2-\nu^2\right)^{\frac{1}{2}}} \frac{\partial}{\partial \xi} \left[\left(\xi^2-1\right)^{\frac{1}{2}} \left(D_p P_p^1(\xi) + E_p Q_p^1(\xi)\right) P_p^1(\nu) \right] \tag{175e}$$

$$B_{\eta III} = - \sum_{p=1}^{\infty} \frac{1}{a\left(\xi^2-\nu^2\right)^{\frac{1}{2}}} \frac{\partial}{\partial \nu} \left[\left(1-\nu^2\right)^{\frac{1}{2}} \left(D_p P_p^1(\xi) + E_p Q_p^1(\xi)\right) P_p^1(\nu) \right] \tag{175f}$$

$$B_{\theta IV} = - \sum_{p=1}^{\infty} \frac{1}{a\left(\xi^2-\nu^2\right)^{\frac{1}{2}}} \frac{\partial}{\partial \xi} \left[\left(\xi^2-1\right)^{\frac{1}{2}}\left(F_p P_p^1(\xi) + G_p Q_p^1(\xi)\right) P_p^1(\nu)\right] \tag{175g}$$

$$B_{\eta IV} = - \sum_{p=1}^{\infty} \frac{1}{a\left(\xi^2-\nu^2\right)^{\frac{1}{2}}} \frac{\partial}{\partial \nu} \left[\left(1-\nu^2\right)^{\frac{1}{2}}\left(F_p P_p^1(\xi) + G_p Q_p^1(\xi)\right) P_p^1(\nu)\right] \tag{175h}$$

$$B_{\theta V} = - \sum_{p=1}^{\infty} \frac{1}{a\left(\xi^2-\nu^2\right)^{\frac{1}{2}}} \frac{\partial}{\partial \xi} \left[\left(\xi^2-1\right)^{\frac{1}{2}} H_p P_p^1(\xi) P_p^1(\nu)\right] \tag{175i}$$

$$B_{\eta V} = - \sum_{p=1}^{\infty} \frac{1}{a\left(\xi^2-\nu^2\right)^{\frac{1}{2}}} \frac{\partial}{\partial \nu} \left[\left(1-\nu^2\right)^{\frac{1}{2}} H_p P_p^1(\xi) P_p^1(\nu)\right] \tag{175j}$$

APPENDIX A

FERROMAGNETIC SPHERICAL BODIES IN A CONSTANT EXTERNAL INDUCING FIELD

INTRODUCTION

In previous work Brown and Baker[7] derived the closed form mathematical expressions for the magnetic flux density for two configurations of a magnetic spherical body surrounded by a stationary current band of azimuthal spherical symmetry. The first case treated was for an infinitesimally thin stationary current band surrounding a spherical magnetic shell. The second case is for a stationary current band of finite width surrounding a solid magnetic sphere. The magnetic bodies were assumed to be linear and homogeneous.

The problem of the magnetic induction for an infinitesimally thin current band surrounding a spherical shell can be generalized to include an external magnetic field. The superposition principle discussed in the text of this report can be used in these two cases to include a constant external magnetic field. The magnetic induction in each region for a three-dimensional magnetic spherical shell in an arbitrary external magnetic field $\overline{H}_o$ is added to the magnetic induction for the corresponding region for the spherical shell surrounded by and/or surrounding a stationary current band. The problem of deriving the magnetic induction for a current band of finite width surrounding a solid ferromagnetic sphere can also be generalized to include an external magnetic field $\overline{H}_o$ in a similar manner. Thus, the magnetic induction for a ferromagnetic spherical body in an external magnetic field must be determined.

Both constant external field problems were solved by Nixon[6] of the Center. The closed form mathematical solutions for the magnetic induction for both constant external field problems were presented in Reference 6 in Cartesian coordinates. It was necessary to convert these mathematical expressions to spherical coordinates to be compatible with this work.

SOLID FERROMAGNETIC SPHERE IN AN EXTERNAL INDUCING FIELD

The solid ferromagnetic sphere in a constant external inducing field is shown in Figure A.1. The permeability of the solid sphere is μ_2 and the radius of the sphere is R_1. The permeability μ_o of vacuum that is external to the sphere is denoted by μ_1. The constant arbitrary magnetic field is designated as $\overline{H}_o$.

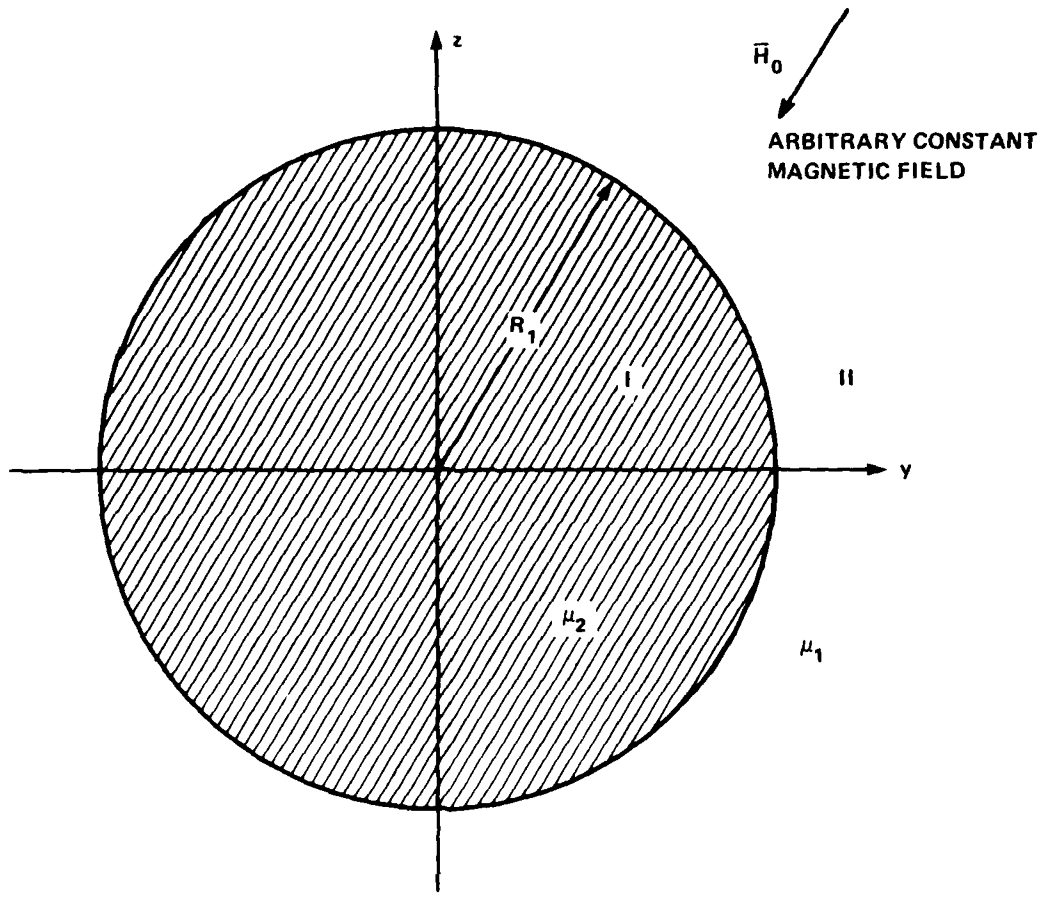


Figure A.1 - Ferromagnetic Spherical Solid in a Constant External Magnetic Field

It is assumed that μ_2 in the sphere is constant, and that μ_1 is constant in the region external to the sphere. Because there are no currents in any region in the problem, the magnetic field $\overline{H}$ can be expressed as the negative of the gradient of a magnetic scalar potential ϕ_m in regions I and II, respectively.

$$\overline{H}_I = -\overline{\nabla}\phi_{Im} \qquad \text{for } 0 \leq r \leq R_1 \tag{A.1a}$$

$$\overline{H}_{II} = -\overline{\nabla}\phi_{IIm} \qquad \text{for } R_1 \leq r < \infty \tag{A.1b}$$

where

$$\overline{B}_I = \mu_2 \overline{H}_I \tag{A.1c}$$

$$\overline{B}_{II} = \mu_1 \overline{H}_{II} \tag{A.1d}$$

The major step toward solving this problem is to determine the solutions of the scalar Laplace's equation in regions I and II which satisfy the boundary conditions at $r = R_1$. In terms of $\overline{B}$ and $\overline{H}$ the magnetostatic boundary conditions are

$$(\overline{B}_{II} - \overline{B}_I) \cdot \overline{n}_{12} = 0 \qquad \text{at } r = R_1 \tag{A.2a}$$

$$\overline{n}_{12} \times (\overline{H}_{II} - \overline{H}_I) = 0 \qquad \text{at } r = R_1 \tag{A.2b}$$

where $\overline{n}_{12}$ is the unit vector normal to the surface of the sphere from region I to II.

The general expression of ϕ_{Im} and ϕ_{IIm} which satisfy Laplace's equation in regions I and II are:

$$\phi_{Im} = Ar \sin\theta \cos\psi + Br \sin\theta \sin\psi + Cr \cos\theta \qquad \text{for } 0 \leq r \leq R_1 \tag{A.3a}$$

$$\phi_{IIm} = Dr^{-2} \sin\theta \cos\psi + Er^{-2} \sin\theta \sin\psi + Fr^{-2} \cos\theta$$

$$- H_{ox} r \sin\theta \cos\psi - H_{oy} r \sin\theta \sin\psi - H_{oz} r \cos\theta$$

$$\text{for } R_1 \le r < \infty \tag{A.3b}$$

The coefficients are determined by the magnetostatic boundary conditions (see Equations (A.2a) and (A.2b) and are:

$$A = - \frac{3\mu_1 H_{ox}}{\mu_2 + 2\mu_1} \tag{A.4a}$$

$$B = - \frac{3\mu_1 H_{oy}}{\mu_2 + 2\mu_1} \tag{A.4b}$$

$$C = - \frac{3\mu_1 H_{oz}}{\mu_2 + 2\mu_1} \tag{A.4c}$$

$$D = \frac{R_1^3 (\mu_2 - \mu_1) H_{ox}}{(\mu_2 + 2\mu_1)} \tag{A.4d}$$

$$E = \frac{R_1^3 (\mu_2 - \mu_1) H_{oy}}{(\mu_2 + 2\mu_1)} \tag{A.4e}$$

$$F = \frac{R_1^3(\mu_2-\mu_1)H_{oz}}{\mu_2+2\mu_1} \quad \text{(A.4f)}$$

For details of this derivation, the reader should consult the work of Nixon.[6] The authors have transformed his mathematical expressions for the magnetic induction in regions I and II from Cartesian to spherical coordinates. This makes the expressions for the magnetic induction compatible with the present work of this report. These mathematical expressions are in regions I and II:

$$\overline{B}_I(r,\theta,\psi) = -\mu_2\hat{e}_r(A \sin\theta \cos\psi + B \sin\theta \sin\psi + C \cos\theta)$$

$$- \mu_2\hat{e}_\theta(A \cos\theta \cos\psi + B \cos\theta \sin\psi - C \sin\theta)$$

$$- \mu_2\hat{e}_\psi(-A \sin\psi + B \cos\psi) \quad \text{for } 0 \leq r \leq R_1 \quad \text{(A.5a)}$$

$$B_{II}(r,\theta,\psi) = +\mu_1\hat{e}_r(2Dr^{-3}\sin\theta \cos\psi + 2\,Er^{-3} \sin\theta \sin\psi + 2\,Fr^{-3} \cos\theta$$

$$+ H_{ox} \sin\theta \cos\psi + H_{oy} \sin\theta \sin\psi + H_{oz} \cos\theta)$$

$$- \mu_1\hat{e}_\theta(Dr^{-3} \cos\theta \cos\psi + Er^{-3} \cos\theta \sin\psi - Fr^{-3} \sin\theta$$

Note: Above equation continued on next page.

$$- H_{ox} \cos\theta \cos\psi - H_{oy} \cos\theta \sin\psi + H_{oz} \sin\theta)$$

$$- \mu_1 \widehat{e}_\psi (-Dr^{-3} \sin\psi + Er^{-3} \cos\psi + H_{ox} \sin\psi - H_{oy} \cos\psi) \tag{A.5b}$$

where
$$\overline{B} = - \mu \overline{\nabla} \phi_m \tag{A.5c}$$

and
$$\nabla \equiv \widehat{e}_r \frac{\partial \phi_m}{\partial r} + \frac{\widehat{e}_\theta}{r} \frac{\partial \phi_m}{\partial \theta} + \frac{\widehat{e}_\psi}{r \sin\theta} \frac{\partial \phi_m}{\partial \psi} \tag{A.5d}$$

FERROMAGNETIC SPHERICAL SHELL IN AN EXTERNAL INDUCING FIELD

The problem of the spherical shell is similar to the problem of the solid sphere. The inner radius of the spherical shell is R_1 and the outer radius is R_2 (see Figure A.2). The permeability of the magnetic material in the shell is μ_2 and the permeability μ_o of vacuum that is internal and external to the shell is denoted by μ_1. The constant external magnetic field is designated by $\overline{H}_o$.

The problem of deriving the closed form mathematical expressions for the magnetic flux density in each of the three regions (I through III) was worked out in detail by Nixon. The problem was solved in a method exactly analogous to the problem of a solid sphere in a constant external magnetic field. For details of the derivation consult Reference 6.

The general expressions for the magnetic scalar potential ϕ_m in regions I through III are:

$$\phi_{Im} = Ar \sin\theta \cos\psi + Br \sin\theta \sin\psi + Cr \cos\theta \qquad \text{for } 0 \leq r \leq R_1 \tag{A.6a}$$

$$\phi_{IIm} = (Gr+Hr^{-2}) \sin\theta \cos\psi + (Ir+Jr^{-2}) \sin\theta \sin\psi$$

$$+ (Kr+Lr^{-2}) \cos\theta \qquad \text{for } R_1 \leq r \leq R_2 \tag{A.6b}$$

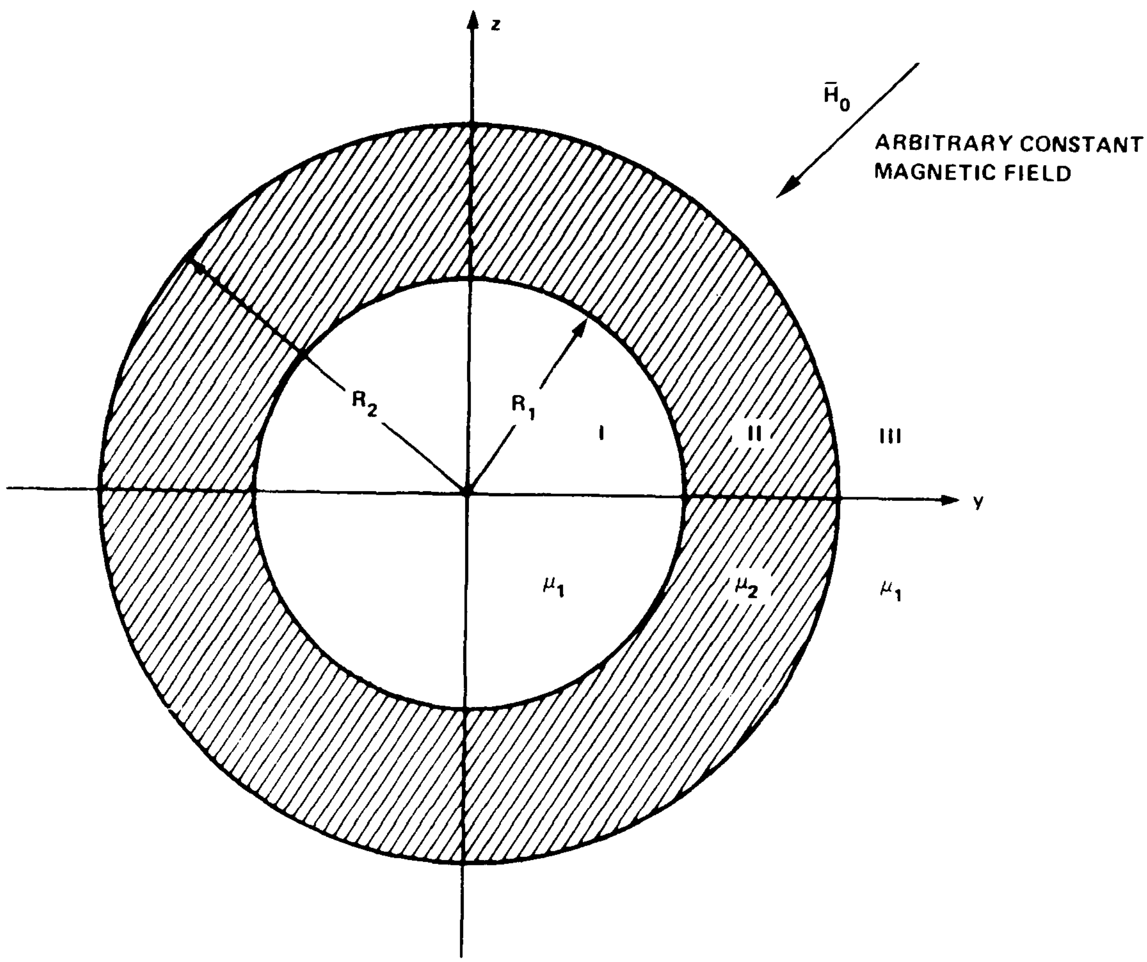


Figure A.2 - Ferromagnetic Spherical Shell in Constant External Magnetic Field

$$\phi_{IIIm} = Dr^{-2} \sin\theta \cos\psi + Er^{-2} \sin\theta \sin\psi + Fr^{-2} \cos\theta$$

$$- H_{ox}\, r \sin\theta \cos\psi - H_{oy}\, r \sin\theta \sin\psi$$

$$- H_{oz}\, r \cos\theta \qquad \text{for } R_2 \leq r < \infty \tag{A.6c}$$

The coefficients are determined by the usual magnetostatic boundary conditions on the spherical surfaces at $r = R_1$ and $r = R_2$. The coefficients determined by this method are:

$$H = -3\mu_1 \left[\frac{(2\mu_1+\mu_2)(2\mu_2+\mu_1)}{(\mu_2-\mu_1)R_1^3} + \frac{2(\mu_1-\mu_2)}{R_2^3} \right]^{-1} H_{ox} \tag{A.7a}$$

$$G = \frac{(2\mu_2+\mu_1)\ H}{(\mu_2-\mu_1)R_1^3} \tag{A.7b}$$

$$D = GR_2^3 + H + H_{ox}\ R_2^3 \tag{A.7c}$$

$$A = G + HR_1^{-3} \tag{A.7d}$$

$$J = -3\mu_1 \left[\frac{(2\mu_1+\mu_2)(2\mu_2+\mu_1)}{(\mu_2-\mu_1)R_1^3} + \frac{2(\mu_1-\mu_2)}{R_2^3} \right]^{-1} H_{oy} \tag{A.7e}$$

$$I = \frac{(2\mu_2+\mu_1)}{(\mu_2-\mu_1)\ R_1^3}\ J \tag{A.7f}$$

$$E = IR_2^3 + J + H_{oy}\ R_2^3 \tag{A.7g}$$

$$B = I + JR_1^{-3} \tag{A.7h}$$

$$L = -3\mu_1 \left[\frac{(2\mu_1+\mu_2)(2\mu_2+\mu_1)}{(\mu_2-\mu_1)\ R_1^3} + \frac{(2\mu_1-\mu_2)}{R_2^3}\right]^{-1} H_{oz} \tag{A.7i}$$

$$K = \frac{(2\mu_2+\mu_1)}{(\mu_2-\mu_1)\ R_1^3}\ L \tag{A.7j}$$

$$F = KR_2^3 + L + H_{oz}\ R_2^3 \tag{A.7k}$$

$$C = K + LR_1^{-3} \tag{A.7ℓ}$$

The authors have transformed Nixon's[6] mathematical expressions for the magnetic induction in regions I through III from Cartesian to spherical coordinates. This makes the expressions for the magnetic induction compatible with the present work of this report. These mathematical expressions in regions I through III are:

$$\overline{B}_I(r,\theta,\psi) = -\mu_1\widehat{e}_r(A \sin\theta \cos\psi + B \sin\theta \cos\psi + C \cos\theta)$$

$$-\mu_1\widehat{e}_\theta(A \cos\theta \cos\psi + B \cos\theta \cos\psi - C \sin\theta)$$

$$-\mu_1\widehat{e}_\psi\ (-A \sin\psi + B \cos\psi) \quad \text{for } 0 \leq r \leq R_1 \qquad \text{(A.8a)}$$

$$\overline{B}_{II}(r,\theta,\psi) = -\mu_2\widehat{e}_r\Big[(G-2r^{-3}H) \sin\theta \cos\psi$$

$$+ (I-2r^{-3}J) \sin\theta \sin\psi + (K-2r^{-3}L) \cos\theta\Big]$$

$$-\mu_2\widehat{e}_\theta\Big[(G+Hr^{-3}) \cos\theta \cos\psi + (I+Jr^{-3})$$

$$(\cos\theta \sin\psi) - (K+Lr^{-3}) \sin\theta\Big]$$

$$-\mu_2\widehat{e}_\psi\Big[-(G+Hr^{-3}) \sin\psi + (I+Jr^{-3}) \cos\psi\Big]$$

$$\text{for } R_1 \leq r \leq R_2 \qquad \text{(A.8b)}$$

$$\overline{B}_{III}(r,\theta,\psi) = + \mu_1\widehat{e}_r \Big[2Dr^{-3} \sin\theta \cos\psi + 2Er^{-3} \sin\theta \sin\psi$$

$$+ 2Fr^{-3} \cos\theta + H_{ox} \sin\theta \cos\psi + H_{oy} \sin\theta \sin\psi + H_{oz} \cos\theta\Big]$$

$$- \mu_1\widehat{e}_\theta\Big[Dr^{-3} \cos\theta \cos\psi + Er^{-3} \cos\theta \sin\psi - Fr^{-3} \sin\theta$$

$$- H_{ox} \cos\theta \cos\psi - H_{oy} \cos\theta \sin\psi + H_{oz} \sin\theta\Big]$$

$$- \mu_1\widehat{e}_\psi\Big[- Dr^{-3} \sin\psi + Er^{-3} \cos\psi + H_{ox} \sin\psi - H_{oy} \cos\psi\Big] \qquad \text{(A.8c)}$$

where

$$\overline{B} = - \mu\overline{\nabla}\phi_m \qquad \text{(A.8d)}$$

and

$$\overline{\nabla} \equiv \widehat{e}_r \frac{\partial\phi_m}{\partial r} + \frac{\widehat{e}_\theta}{r} \frac{\partial\phi_m}{\partial\theta} + \frac{\widehat{e}_\psi}{r \sin\theta} \frac{\partial\phi_m}{\partial\psi} \qquad \text{(A.8e)}$$

APPENDIX B

DERIVATION OF THE COEFFICIENTS OF THE VECTOR POTENTIAL FOR A SOLID SPHERE SURROUNDED BY AN INFINITESIMALLY THIN SPHERICAL CURRENT BAND AND REDUCTION OF THE MAGNETIC VECTOR POTENTIAL TO THAT OF A THIN BAND IN VACUUM WHEN IN THE LIMIT μ_2 EQUALS μ_1

DERIVATION OF THE COEFFICIENTS

In this appendix the coefficients are derived for the vector potential in regions I through III for a ferromagnetic sphere surrounded by an infinitely thin current band. For a detailed discussion of this ferromagnetic problem, see the section in the text of the report entitled "Solid Sphere Surrounded by an Infinitesimally Thin Spherical Current Band". The magnetic vector potential in each region is given by:

$$A_I = A_{\psi I} = \sum_{p=1}^{\infty} \left(A_{p1} r^p\right) P_p^1(\cos\theta) \tag{B.1a}$$

$$A_{II} = A_{\psi II} = \sum_{p=1}^{\infty} \left[A_{p2} r^p + \frac{B_{p2}}{r^{(p+1)}}\right] P_p^1(\cos\theta) \tag{B.1b}$$

$$A_{III} = A_{\psi III} = \sum_{p=1}^{\infty} \left[\frac{B_{p3}}{r^{(p+1)}}\right] P_p^1(\cos\theta) \tag{B.1c}$$

The coefficients (A_{pi}) and (B_{pi}) in Equations (B.1a) through (B.1c) are obtained by substituting these equations into the boundary conditions (Equations (B.2a) through (B.2d)).

$$A_I = A_{II} \qquad \text{at } r = R_1 \tag{B.2a}$$

$$A_{II} = A_{III} \qquad \text{at } r = R_2 \tag{B.2b}$$

$$-\frac{1}{\mu_1}\;\frac{1}{r}\;\frac{\partial}{\partial r}\,(rA_{II}) + \frac{1}{\mu_2}\;\frac{1}{r}\;\frac{\partial}{\partial r}\,(rA_I) = 0 \quad \text{at } r = R_1 \tag{B.2c}$$

$$-\frac{1}{\mu_1}\;\frac{1}{r}\;\frac{\partial}{\partial r}\,(rA_{III}) + \frac{1}{\mu_1}\;\frac{1}{r}\;\frac{\partial}{\partial r}\,(rA_{II}) = J(\theta) \quad \text{at } r = R_2 \tag{B.2d}$$

After appropriate substitutions of Equations (B.1a) through (B.1c) into Equations (B.2a) through (B.2d), the following boundary value equations are obtained.

$$A_{p1}R_1{}^p = \left[A_{p2}R_1{}^p + B_{p2}R_1{}^{-(p+1)}\right] \tag{B.3a}$$

$$\left[A_{p2}R_2{}^p + B_{p2}R_2{}^{-(p+1)}\right] = B_{p3}R_2{}^{-(p+1)} \tag{B.3b}$$

$$-\frac{1}{\mu_1}\left[A_{p2}(p+1)R_1^{(p-1)} - B_{p2}pR_1^{-(p+2)}\right] + \frac{1}{\mu_2}\left[A_{p1}(p+1)R_1^{(p-1)}\right] = 0 \tag{B.3c}$$

$$\left[\frac{1}{\mu_1}\;B_{p3}pR_2{}^{-(p+2)}\right] + \left[\frac{1}{\mu_1}\right]\left[A_{p2}(p+1)R_2{}^{p-1} - pB_{p2}R_2{}^{-(p+2)}\right] = \frac{J_p(\theta)}{P_p^1(\cos\theta)} \tag{B.3d}$$

These algebraic equations provide four simultaneous equations, with four unknowns, which can be solved for the coefficients A_{pi} and B_{pi} by algebraic manipulation.

Solving Equation (B.3a) for A_{p1} and Equation (B.3b) for B_{p3} we have

$$A_{p1} = A_{p2} + B_{p2}R_1^{-(2p+1)} \tag{B.4}$$

$$B_{p3} = \frac{A_{p2}R_2^p + B_{p2}R_2^{-(p+1)}}{R_2^{-(p+1)}} \tag{B.5}$$

Solving Equation (B.3c) for A_{p1} gives

$$A_{p1} = \left[\frac{\mu_2}{\mu_1}\right]\left[A_{p2} - \frac{B_{p2}pR_1^{-(2p+1)}}{(p+1)}\right] \tag{B.6}$$

which, when substituted into Equation (B.3a) and solved for A_{p2}, yields

$$A_{p2} = B_{p2}\,\frac{\left[-R_1^{-(2p+1)} - \left(\frac{\mu_2}{\mu_1}\right)\left(\frac{pR_1{}^{-(2p+1)}}{(p+1)}\right)\right]}{\left(1 - \frac{\mu_2}{\mu_1}\right)} \tag{B.7}$$

Substituting Equations (B.5) and (B.7) into Equation (B.3d) gives the expression for B_{p2}

$$B_{p2} = \frac{\left(1 - \frac{\mu_2}{\mu_1}\right)\left(\frac{\mu_1 J_p(\theta)}{P_p^1(\cos\theta)}\right)}{\left(-R_1^{-(2p+1)}\right)(2p+1)R_2^{p-1}\left(1 + \left(\frac{\mu_2}{\mu_1}\right)\frac{p}{(p+1)}\right)} \tag{B.8}$$

Simplifying Equation (B.7) by using Equation (B.8) yields

$$A_{p2} = \frac{\mu_1 J_p(\theta)}{(2p+1)\ R_2^{(p-1)} P_p^1(\cos\theta)} \tag{B.9}$$

The following expression is obtained for A_{p1} from Equation (B.4) after substituting Equations (B.8) and (B.9) for B_{p2} and A_{p2}, respectively, as

$$A_{p1} = \frac{\mu_1 J_p(\theta)}{(2p+1) R_2^{(p-1)} P_p^1(\cos\theta)} - \frac{\left[\left(1 - \frac{\mu_2}{\mu_1}\right) \frac{\mu_1 J_p(\theta)}{P_p^1(\cos\theta)}\right]}{(2p+1) R_2^{(p-1)} \left(1 + \left(\frac{\mu_2}{\mu_1}\right) \frac{p}{(p+1)}\right)} \tag{B.10}$$

The mathematical solution for B_{p3} in terms of known quantities is obtained from Equation (B.5) by substituting the previously obtained expressions for A_{p2} (Equation (B.9)) and for B_{p2} (Equation (B.8)).

$$B_{p3} = \frac{\frac{\mu_1 J_p(\theta) R_2^{(2p+1)}}{P_p^1(\cos\theta)}}{(2p+1) R_2^{(p-1)}} - \frac{\left(1 - \frac{\mu_2}{\mu_1}\right)\left(\frac{\mu_1 J_p(\theta)}{P_p^1(\cos\theta)}\right)}{\left(R_1^{-(2p+1)} \right) (2p+1) R_2^{(p-1)} \left(1 + \frac{\mu_2}{\mu_1} \frac{p}{(p+1)}\right)} \tag{B.11}$$

After the numerical value for B_{p3} is calculated on the computer for a specific problem, the numerical values for the other coefficients can be obtained from the following equations:

$$B_{p2} = \frac{\left(1 - \frac{\mu_2}{\mu_1}\right) \frac{\mu_1 J_p(\theta)}{P_p^1(\cos\theta)}}{\left(-R^{-(2p+1)}\right) \; (2p+1) R_2^{(p-1)} \left(1 + \left(\frac{\mu_2}{\mu_1}\right) \frac{p}{p+1}\right)} \qquad \text{(B.12a)}$$

$$A_{p2} = \frac{\frac{\mu_1 J_p(\theta)}{P_p^1(\cos\theta)}}{(2p+1) R_2^{(p-1)}} \qquad \text{(B12.b)}$$

$$A_{p1} = \frac{\mu_1 J_p(\theta)}{(2p+1) R_2^{(p-1)} P_p^1(\cos\theta)} - \frac{\left(1 - \frac{\mu_2}{\mu_1}\right) \frac{\mu_1 J_p(\theta)}{P_p^1(\cos\theta)}}{(2p+1) R_2^{p-1} \left(1 + \left(\frac{\mu_2}{\mu_1}\right) \frac{p}{p+1}\right)} \qquad \text{(B.12c)}$$

REDUCTION OF THE POTENTIALS WHEN μ_2 EQUALS μ_1

The vector potentials for this problem should reduce to those of an infinitesimally thin current band in a homogeneous medium with permeability μ_1 in the limit as $\mu_2 = \mu_1$ (see Figure B.1). In this limit the coefficients should assume the following form

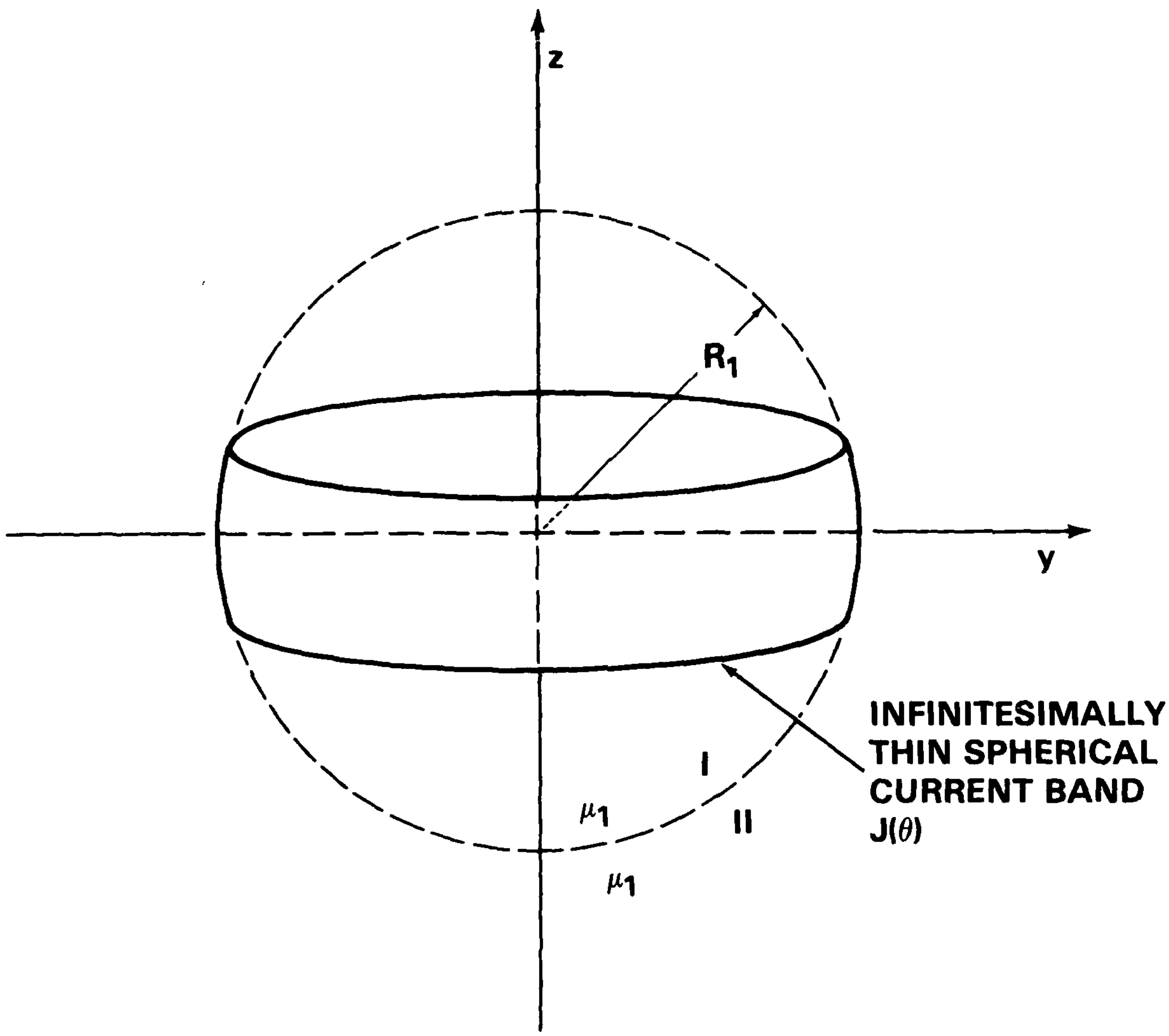


Figure B.1 - Infinitesimally Thin Current Band

$$A_{p1} = A_{p2};\ B_{p2} = 0;\ B_{p3} \neq 0 \tag{B.13}$$

where A_{p1} and B_{p3} should reduce to the coefficients for the potentials in the two regions for the spherical band problem of Appendix B in Reference 7. One immediately observes, from Equation (B.12a) that $B_{p2} = 0$ when $\mu_2 = \mu_1$. From Equations (B.12b) and (B.12c) we see that

$$A_{p2} = \frac{\left[\dfrac{\mu_1 J_p(\theta)}{P_p^1(\cos\theta)}\right]}{(2p+1)R_2^{(p-1)}} \tag{B.14a}$$

$$A_{p1} = A_{p2} \tag{B.14b}$$

From Equation (B.11), in the limit of $\mu_2 = \mu_1$, we have

$$B_{p3} = \frac{\mu_1 J_p(\theta)}{R_2^{-(p+2)}(2p+1)P_p^1(\cos\theta)} \tag{B.15}$$

Rewriting A_{p1} we have

$$A_{p1} = B_{p3}\ R_2^{-(2p+1)} \tag{B.16}$$

This means that in the three regions the components of the vector potentials used in Equations (B.1):

$$A_{\psi I} = \sum_{p=1}^{\infty} \left(A_{p1} r^{P}\right) P_p^1(\cos\theta) \tag{B.17a}$$

$$A_{\psi II} = \sum_{p=1}^{\infty} \left[A_{p2} r^{P} + \frac{B_{p2}}{r^{(p+1)}}\right] P_p^1(\cos\theta) \tag{B.17b}$$

$$A_{\psi III} = \sum_{p=1}^{\infty} \left[\frac{B_{p3}}{r^{(p+1)}}\right] P_p^1(\cos\theta) \tag{B.17c}$$

reduce, when $\mu_2 = \mu_1$, to the form

$$A_{\psi I} = A_{\psi II} = \sum_{p=1}^{\infty} \left[\left(A_{p1}\Big|_{\mu_2=\mu_1}\right) r^{P}\right] P_p^1(\cos\theta) \tag{B.18a}$$

$$A_{\psi III} = \sum_{p=1}^{\infty} \left[\left(B_{p3}\Big|_{\mu_2=\mu_1}\right) r^{-(p+1)}\right] P_p^1(\cos\theta) \tag{B.18b}$$

The mathematical expressions for A_{p1} and B_{p3} (see Equations (B.16) and B.15), respectively), for the ferromagnetic sphere surrounded by a thin current band in the limit as $\mu_2 = \mu_1$, are the same as for the coefficients A_{p1} and B_{p2} (see Reference 7, Appendix B), respectively, for the components of the vector potentials in the regions of the current band in vacuum. For comparison, the coefficients for the current band problem are:

$$A_{p1} = B_{p2} R_1^{-(2p+1)} \tag{B.19a}$$

$$B_{p2} = \frac{\mu_1 J_p(\theta)}{P_p^1(\cos\theta) R_1^{-(p+2)}(2p+1)} \tag{B.19b}$$

and the coefficients for the ferromagnetic sphere problem with $\mu_2 = \mu_1$ are

$$A_{p1} = B_{p3} R_2^{-(2p+1)} \tag{B.20a}$$

$$B_{p3} = \frac{\mu_1 J_p(\theta)}{P_p^1(\cos\theta) R_2^{-(p+2)}(2p+1)} \tag{B.20b}$$

It is noted when making the comparison, R_2 must be set equal to R_1.

APPENDIX C

DERIVATION OF THE COEFFICIENTS OF THE VECTOR POTENTIAL FOR AN INFINITESIMALLY THIN CURRENT BAND SURROUNDED BY A FERROMAGNETIC SPHERICAL SHELL AND THE REDUCTION OF THE MAGNETIC VECTOR POTENTIAL TO THAT OF A THIN BAND IN VACUUM WHEN IN THE LIMIT μ_2 EQUALS μ_1

DERIVATION OF THE COEFFICIENTS

In this appendix the coefficients are derived for the vector potential in regions I through IV for a ferromagnetic sphere surrounding an infinitely thin current band. For a detailed discussion of this ferromagnetic problem, see the section in the text of the report entitled, "Hollow Sphere Surrounding an Infinitesimally Thin Spherical Current Band". The magnetic vector potential in each region is given by:

$$A_I = A_{\psi I} = \sum_{p=1}^{\infty} \left[A_{p1} r^p \right] P_p^1(\cos\theta) \tag{C.1a}$$

$$A_{II} = A_{\psi II} = \sum_{p=1}^{\infty} \left[A_{p2} r^p + \frac{B_{p2}}{r^{(p+1)}} \right] P_p^1(\cos\theta) \tag{C.1b}$$

$$A_{III} = A_{\psi III} = \sum_{p=1}^{\infty} \left[A_{p3} r^p + \frac{B_{p3}}{r^{(p+1)}} \right] P_p^1(\cos\theta) \tag{C.1c}$$

$$A_{IV} = A_{\psi IV} = \sum_{p=1}^{\infty} \left[\frac{B_{p4}}{r^{(p+1)}} \right] P_p^1(\cos\theta) \tag{C.1d}$$

The coefficients (A_{pl} and B_{pl}) in Equations (C.1a through C.1d) are obtained by substituting these equations into the boundary conditions (Equations (C.2a) through (C.2f)).

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$$A_{I} = A_{II} \qquad r = R_1 \tag{C.2a}$$

$$A_{II} = A_{III} \qquad r = R_2 \tag{C.2b}$$

$$A_{III} = A_{IV} \qquad r = R_3 \tag{C.2c}$$

$$-\frac{1}{\mu_1}\,\frac{1}{r}\,\frac{\partial}{\partial r}(rA_{II}) + \frac{1}{\mu_1}\,\frac{1}{r}\,\frac{\partial}{\partial r}(rA_{I}) = J(\theta) \qquad r = R_1 \tag{C.2d}$$

$$-\frac{1}{\mu_2}\,\frac{1}{r}\,\frac{\partial}{\partial r}(rA_{III}) + \frac{1}{\mu_1}\,\frac{1}{r}\,\frac{\partial}{\partial r}(rA_{II}) = 0 \qquad r = R_2 \tag{C.2e}$$

$$-\frac{1}{\mu_1}\,\frac{1}{r}\,\frac{\partial}{\partial r}(rA_{IV}) + \frac{1}{\mu_2}\,\frac{1}{r}\,\frac{\partial}{\partial r}(rA_{III}) = 0 \qquad r = R_3 \tag{C.2f}$$

After appropriate substitutions of Equations (C.1a) through (C.1d) into Equations (C.2a) through (C.2f), the following boundary value equations are obtained.

$$A_{p1}R_1^p = A_{p2}R_1^p + B_{p2}R_1^{-(p+1)} \tag{C.3a}$$

$$A_{p2}R_2^p + B_{p2}R_2^{-(p+1)} = A_{p3}R_2^p + B_{p3}R_2^{-(p+1)} \tag{C.3b}$$

$$A_{p3}R_3^p + B_{p3}R_3^{-(p+1)} = B_{p4}R_3^{-(p+1)} \tag{C.3c}$$

$$- \frac{1}{\mu_1}\left[A_{p2}(p+1)R_1^{(p-1)} - pB_{p2}R_1^{-(p+2)}\right] + \frac{1}{\mu_1}\left[A_{p1}(p+1)R_1^{(p-1)}\right] = \frac{J_p(\theta)}{P_p^1(\cos\theta)} \quad \text{(C.3d)}$$

$$- \frac{1}{\mu_2}\left[A_{p3}(p+1)R_2^{(p-1)} - pB_{p3}R_2^{-(p+2)}\right] + \frac{1}{\mu_1}\left[A_{p2}(p+1)R_2^{(p-1)} - pB_{p2}R_2^{-(p+2)}\right] = 0 \quad \text{(C.3e)}$$

$$\frac{1}{\mu_1}\left[pB_{p4}R_3^{-(p+2)}\right] + \frac{1}{\mu_2}\left[A_{p3}(p+1)R_3^{(p-1)} - pB_{p3}R_3^{-(p+2)}\right] = 0 \quad \text{(C.3f)}$$

The algebraic equations provide six simultaneous equations with six unknowns to which can be solved for the coefficients A_{pi} and B_{pi}, by algebraic manipulation.

Solving Equation (C.3a) and Equation (C.3d) for A_{p1}, respectively, we have

$$A_{p1} = A_{p2} + B_{p2}R_1^{-(2p+1)} \quad \text{(C.4a)}$$

$$A_{p1} = \frac{\mu_1 J_p(\theta)}{P_p^1(\cos\theta)(p+1)} R_1^{-(p-1)} + A_{p2} - \frac{B_{p2}pR_1^{-(2p+1)}}{(p+1)} \quad \text{(C.4b)}$$

Equating (C.4a) and (C.4b) and solving for B_{p2} yields

$$B_{p2} = \frac{\mu_1 J_p(\theta)R_1^{p+2}}{(2p+1)P_p^1(\cos\theta)} = J_p'(\theta) \quad \text{(C.5)}$$

Using Equation (C.3c) to solve for B_{p4} we have

$$B_{p4} = A_{p3} R_3^{(2p+1)} + B_{p3} \tag{C.6}$$

and similarly, using Equation (C.3f) to solve for B_{p4}, we have

$$B_{p4} = -\frac{\mu_1}{\mu_2} \left[A_{p3} \left(\frac{(p+1)}{p} \right) R_3^{(2p+1)} - B_{p3} \right] \tag{C.7}$$

Now, upon equating Equations (C.6) and (C.7) and solving for B_{p3}, we obtain

$$B_{p3} = A_{p3} \left[X \right] \tag{C.8}$$

where
$$\left[X \right] = \frac{-R_3^{(2p+1)} \left\{ 1 + \frac{\mu_1}{\mu_2} \left(\frac{p+1}{p} \right) \right\}}{\left(1 - \frac{\mu_1}{\mu_2} \right)}$$

Solving Equation (C.3b) for A_{p2} and substituting Equation (C.5) for B_{p2} yields

$$A_{p2} = -J_p'(\theta) R_2^{-(2p+1)} + A_{p3} + B_{p3} R_2^{-(2p+1)} \tag{C.9}$$

Using Equations (C.5), (C.8), and (C.9) in Equation (C.3e) yields the following expression for A_{p3}

$$A_{p3} = \frac{\frac{1}{\mu_1}\left[J_p'(\theta)(2p+1)R_2^{-(p+2)}\right]}{\left[\frac{1}{\mu_1}(p+1)R_2^{(p-1)} + \frac{1}{\mu_1}\left[X\right](p+1)R_2^{-(p+2)} - \frac{1}{\mu_2}(p+1)R_2^{(p-1)} + \frac{1}{\mu_2}p\left[X\right]R_2^{-(p+2)}\right]} \tag{C.10}$$

where

$$\left[X\right] = \frac{-R_3^{(2p+1)}\left\{1 + \frac{\mu_1}{\mu_2}\left(\frac{p+1}{p}\right)\right\}}{\left(1 - \frac{\mu_1}{\mu_2}\right)}$$

$$J_p'(\theta) = \frac{\mu_1 J_p(\theta) R_1^{p+2}}{(2p+1)P_p^1(\cos\theta)}$$

The constants have now been found. After the numerical value of A_{p3} is calculated for a specific problem, the numerical values for the other coefficients can be obtained from the following equations:

$$B_{p2} = J_p'(\theta) \tag{C.11a}$$

$$B_{p3} = A_{p3}\left[X\right] \tag{C.11b}$$

$$A_{p2} = -J_p'(\theta)R_2^{-(2p+1)} + A_{p3} + B_{p3}R^{-(2p+1)} \tag{C.11c}$$

$$A_{p1} = A_{p2} + B_{p2}R_1^{-(2p+1)} \tag{C.11d}$$

$$B_{p4} = A_{p3} R_3^{(2p+1)} + B_{p3} \tag{C.11e}$$

REDUCTION OF THE POTENTIAL WHEN μ_2 EQUALS μ_1

When μ_1 is set equal to μ_2 the above ferromagnetic problem reduces to that of finding the potentials in the two regions of a simple current band (see Figure (B.1), because the ferromagnetic shell will now have a permeability μ_1 equal to that of the homogeneous medium with a permeability μ_1. In this limit the coefficients should assume the following form:

$$\begin{array}{ll} A_{p1} \neq 0 & B_{p2} = B_{p4} \\ A_{p2} = 0 & B_{p3} = B_{p4} \\ A_{p3} = 0 & B_{p4} \neq 0 \end{array} \tag{C.12}$$

where A_{p1} and B_{p4} should reduce to the coefficients for the potentials in the two regions for the spherical band problem (see Reference 7, Appendix B). If the coefficients assume this mathematical form it will prove that the mathematical forms of the coefficients for the spherical shell surrounding a thin current band are mathematically correct.

The coefficient A_{p3} will now be evaluated when the limit is taken with $\mu_2 = \mu_1$ which causes $\left[X\right]$ to approach infinity (see Equations (C.10)).

$$A_{p3}\Bigg|_{\mu_2=\mu_1} = \lim_{\left[X\right]\to\infty} \left\{ \frac{\frac{1}{\mu_1} J_p'(2p+1) R_2^{-(p+2)}}{\frac{1}{\mu_1}(p+1) R_2^{(p-1)} + \frac{1}{\mu_1}\left[X\right](p+1) R_2^{-(p+2)} - \frac{1}{\mu_1}(p+1) R_2^{(p-1)} + \frac{1}{\mu_1} p \left[X\right] R_2^{-(p+2)}} \right\} \tag{C.13}$$

$$A_{p3}\Big|_{\mu_2=\mu_1} = 0$$

where
$$J_p' = \frac{\mu_1 J_p(\theta)}{(2p+1)R_I^{-(p+2)}P_p^1(\cos\theta)}$$

The expression for B_{p2} (see Equation (C.11a)) is

$$B_{p2} = J_p' \tag{C.14}$$

The expression for B_{p2} when $\mu_2 = \mu_1$ is still J_p' since J_p' is not a function of $\left[X\right]$

$$B_{p2}\Big|_{\mu_2=\mu_1} = J_p' \tag{C.15}$$

The expression for B_{p3} (see Equation (C.11b)) is

$$B_{p3} = A_{p3}\left[X\right] \tag{C.16}$$

where $\mu_2 = \mu_1$, B_{p3} can be written as

$$B_{p3}\Big|_{\mu_2=\mu_1} = \underset{[X]\to\infty}{\ell\text{imit}} \left\{ \frac{[X]\frac{1}{\mu_1} J_p'(2p+1)R_2^{-(p+2)}}{\frac{1}{\mu_1}(p+1)R_2^{(p-1)} + \frac{1}{\mu_1}[X](p+1)R_2^{-(p+2)} - \frac{1}{\mu_1}(p+1)R_2^{(p-1)} + \frac{1}{\mu_1}p[X]R_2^{-(p+2)}} \right\}$$

$$= \underset{[X]\to\infty}{\ell\text{imit}} \left\{ \frac{\frac{1}{\mu_1}[X]J_p'(2p+1)R_2^{-(p+2)}}{\frac{1}{\mu_1}[X](2p+1)R_2^{-(p+2)}} \right\} \tag{C.17}$$

$$B_{p3}\Big|_{\mu_2=\mu_1} = J_p'$$

The expression for A_{p2} (see Equation (C.11c) is

$$A_{p2} = -J_p'R_2^{-(2p+1)} + A_{p3} + B_{p3}R_2^{-(2p+1)} \tag{C.18}$$

when $\mu_2 = \mu_1$, A_{p2} can be written as

$$A_{p2}\Big|_{\mu_2=\mu_1} = -J_p'R^{-(2p+1)} + A_{p3}\Big|_{\mu_2=\mu_1} + \left(B_{p3}\Big|_{\mu_2=\mu_1}\right)R_2^{-(2p+1)} \tag{C.19}$$

$$A_{p2}\Big|_{\mu_2=\mu_1} = A_{p3}\Big|_{\mu_2=\mu_1} = 0 \tag{C.20}$$

The expression for B_{p4} (see Equation (C.11e)) is

$$B_{p4} = A_{p3}R_3^{(2p+1)} + B_{p3} \tag{C.21}$$

when $\mu_2 = \mu_1$, B_{p4} can be written as

$$B_{p4}\Big|_{\mu_2=\mu_1} = \left(A_{p3}\Big|_{\mu_2=\mu_1}\right)R_3^{(2p+1)} + B_{p3}\Big|_{\mu_2=\mu_1} \tag{C.22}$$

$$B_{p4}\Big|_{\mu_2=\mu_1} = B_{p3}\Big|_{\mu_2=\mu_1} = J_p' \tag{C.23}$$

The expression for A_{p1} (see Equation (C.11d)) is

$$A_{p1} = A_{p2} + B_{p2}R_1^{-(2p+1)}$$

$$A_{p1}\Big|_{\mu_2=\mu_1} = A_{p2}\Big|_{\mu_2=\mu_1} + \left(B_{p2}\Big|_{\mu_2=\mu_1}\right)R_1^{-(2p+1)}$$

$$A_{p1}\Big|_{\mu_2=\mu_1} = \left(B_{p2}\Big|_{\mu_2=\mu_1}\right)R_1^{-(2p+1)}$$

$$A_{p1}\Big|_{\mu_2=\mu_1} = \left(B_{p4}\Big|_{\mu_2=\mu_1}\right)R_1^{-(2p+1)}$$

$$A_{p1}\Big|_{\mu_2=\mu_1} = \left(J_p'\right) R_1^{-(2p+1)} \tag{C.24}$$

This means that in the four regions, the components of the vector potentials used in Equation (C.1)

$$A_{\psi I} = \sum_{p=1}^{\infty} \left[A_{p1} r^{p}\right] P_p^1(\cos\theta) \tag{C.25a}$$

$$A_{\psi II} = \sum_{p=1}^{\infty} \left[A_{p2} r^{p} + \frac{B_{p2}}{r^{(p+1)}}\right] P_p^1(\cos\theta) \tag{C.25b}$$

$$A_{\psi III} = \sum_{p=1}^{\infty} \left[A_{p3} r^{p} + \frac{B_{p3}}{r^{(p+1)}}\right] P_p^1(\cos\theta) \tag{C.25c}$$

$$A_{\psi IV} = \sum_{p=1}^{\infty} \left[\frac{B_{p4}}{r^{(p+1)}}\right] P_p^1(\cos\theta) \tag{C.26d}$$

reduce, when $\mu_2 = \mu_1$, to the form

$$A_{\psi I} = \sum_{p=1}^{\infty} \left[\left(A_{p1}\Big|_{\mu_2=\mu_1}\right) r^{p}\right] P_p^1(\cos\theta) \tag{C.26a}$$

$$A_{\psi II,III,IV} = \sum_{p=1}^{\infty} \left[\left(B_{p4}\Big|_{\mu_2=\mu_1}\right) r^{-(p+1)}\right] P_p^1(\cos\theta) \tag{C.26b}$$

The mathematical expressions for A_{p1} and B_{p4} (see Equations (C.24) and (C.23), respectively) for the ferromagnetic spherical shell surrounding a thin current band, in the limit as $\mu_2 = \mu_1$, are the same as the coefficients A_{p1} and B_{p2} (see Reference 7, Appendix B), respectively, for the components of the vector potentials in the regions of the current band in vacuum. For comparison, the coefficients for the current band problem are

$$A_{p1} = B_{p2} R_1^{-(2p+1)} \tag{C.27a}$$

$$B_{p2} = \frac{\mu_1 J_p(\theta)}{P_p^1(\cos\theta) R_1^{-(p+2)}(2p+1)} \tag{C.27b}$$

and the coefficients for the ferromagnetic shell problem with $\mu_2 = \mu_1$ are

$$A_{p1} = B_{p4} R_1^{-(2p+1)} \tag{C.28a}$$

$$B_{p4} = \frac{\mu_1 J_p(\theta)}{P_p^1(\cos\theta) R_1^{-(p+2)}(2p+1)} \tag{C.28b}$$

APPENDIX D

DERIVATION OF THE COEFFICIENTS OF THE VECTOR POTENTIAL FOR A SPHERICAL SHELL WITH INTERNAL AND EXTERNAL INFINITESIMALLY THIN SPHERICAL CURRENT BANDS AND THE REDUCTION OF THE MAGNETIC VECTOR POTENTIAL TO THAT OF A THIN BAND IN VACUUM WHEN IN THE LIMIT μ_2 EQUALS μ_1

REDUCTION OF THE COEFFICIENTS

In this appendix, the coefficients are derived for the vector potential in regions I through V for a ferromagnetic spherical shell with internal and external, infinitely thin, spherical current bands. For a detailed discussion of the ferromagnetic problem, see the body of this report. The components of the magnetic vector potential in each region are given by

$$A_{I} = A_{\psi I} = \sum_{p=1}^{\infty} \left[A_{p1} r^{p}\right] P_p^1(\cos\theta) \tag{D.1a}$$

$$A_{II} = A_{\psi II} = \sum_{p=1}^{\infty} \left[A_{p2} r^{p} + \frac{B_{p2}}{r^{(p+1)}}\right] P_p^1(\cos\theta) \tag{D.1b}$$

$$A_{III} = A_{\psi III} = \sum_{p=1}^{\infty} \left[A_{p3} r^{p} + \frac{B_{p3}}{r^{(p+1)}}\right] P_p^1(\cos\theta) \tag{D.1c}$$

$$A_{IV} = A_{\psi IV} = \sum_{p=1}^{\infty} \left[A_{p4} r^{p} + \frac{B_{p4}}{r^{(p+1)}}\right] P_p^1(\cos\theta) \tag{D.1d}$$

$$A_{V} = A_{\psi V} = \sum_{p=1}^{\infty} \left[\frac{B_{p5}}{r^{(p+1)}}\right] P_p^1(\cos\theta) \tag{D.1e}$$

The coefficients in Equations (D.1a) through (D.1e) are obtained by substituting these equations into boundary condition (Equations (D.2a) through (D.2h).

$$A_I = A_{II} \qquad \text{at } r = R_1 \tag{D.2a}$$

$$A_{II} = A_{III} \qquad \text{at } r = R_2 \tag{D.2b}$$

$$A_{III} = A_{IV} \qquad \text{at } r = R_3 \tag{D.2c}$$

$$A_{IV} = A_V \qquad \text{at } r = R_4 \tag{D.2d}$$

$$-\frac{1}{\mu_1}\frac{1}{r}\frac{\partial}{\partial r}(rA_{II}) + \frac{1}{\mu_1}\frac{1}{r}\frac{\partial}{\partial r}(rA_I) = J_1(\theta) \qquad r = R_1 \tag{D.2e}$$

$$-\frac{1}{\mu_2}\frac{1}{r}\frac{\partial}{\partial r}(rA_{III}) + \frac{1}{\mu_1}\frac{1}{r}\frac{\partial}{\partial r}(rA_{II}) = 0 \qquad r = R_2 \tag{D.2f}$$

$$-\frac{1}{\mu_1}\frac{1}{r}\frac{\partial}{\partial r}(rA_{IV}) + \frac{1}{\mu_2}\frac{1}{r}\frac{\partial}{\partial r}(rA_{III}) = 0 \qquad r = R_3 \tag{D.2g}$$

$$-\frac{1}{\mu_1}\frac{1}{r}\frac{\partial}{\partial r}(rA_V) + \frac{1}{\mu_1}\frac{1}{r}\frac{\partial}{\partial r}(rA_{IV}) = J_2(\theta) \qquad r = R_4 \tag{D.2h}$$

After the appropriate substitutions are made, the following equations are obtained:

$$A_{p1}R_1^p = \left[A_{p2}R_1^p + B_{p2}R_1^{-(p+1)}\right] \tag{D.3a}$$

$$\left[A_{p2}R_2^p + B_{p2}R_2^{-(p+1)}\right] = \left[A_{p3}R_2^p + B_{p3}R_2^{-(p+1)}\right] \tag{D.3b}$$

$$\left[A_{p3}R_3^p + B_{p3}R_3^{-(p+1)}\right] = \left[A_{p4}R_3^p + B_{p4}R_3^{-(p+1)}\right] \tag{D.3c}$$

$$\left[A_{p4}R_4^p + B_{p4}R_4^{-(p+1)}\right] = \left[B_{p5}R_4^{-(p+1)}\right] \tag{D.3d}$$

$$-\frac{1}{\mu_1}\left[A_{p2}(p+1)R_1^{(p-1)} - pB_{p2}R_1^{-(p+2)}\right] + \frac{1}{\mu_1}\left[(p+1)A_{p1}R_1^{(p-1)}\right] = \frac{J_{p1}(\theta)}{P_p^1(\cos\theta)} \tag{D.3e}$$

$$-\frac{1}{\mu_2}\left[A_{p3}(p+1)R_2^{(p-1)} - pB_{p3}R_2^{-(p+2)}\right] + \frac{1}{\mu_1}\left[A_{p2}(p+1)R_2^{(p-1)} - pB_{p2}R_2^{-(p+2)}\right] = 0 \tag{D.3f}$$

$$-\frac{1}{\mu_1}\left[A_{p4}(p+1)R_3^{(p-1)} - pB_{p4}R_3^{-(p+2)}\right] + \frac{1}{\mu_2}\left[A_{p3}(p+1)R_3^{(p-1)} - pB_{p3}R_3^{-(p+2)}\right] = 0 \tag{D.3g}$$

$$\frac{1}{\mu_1}\left[pR_4^{-(p+2)}B_{p5}\right] + \frac{1}{\mu_1}\left[A_{p4}(p+1)R_4^{(p-1)} - pB_{p4}R_4^{-(p+2)}\right] = \frac{J_{p2}(\theta)}{P_p^1(\cos\theta)} \tag{D.3h}$$

These algebraic equations provide eight simultaneous equations with eight unknowns, which can be solved for the coefficients A_{p4} and B_{p4} by algebraic manipulation.

Solving Equations (D.3h) and (D.3d) for B_{p4} and equating the results to solve for A_{p4}, yields

$$A_{p4} = \frac{\mu_1 J_{p2}(\theta)}{(2p+1) R_4^{(p-1)} P_p^1(\cos\theta)} \equiv \left[X\right] \tag{D.4}$$

Solving Equations (D.3b) and (D.3f) for B_{p3} and equating the results to solve for A_{p2}, yields

$$A_{p2} = \frac{A_{p3}\left(\frac{(2p+1)}{p}\right) R_2^{(2p+1)} + \left(\frac{\mu_2}{\mu_1} - 1\right) B_{p2}}{\left[1 + \frac{\mu_2}{\mu_1}\left(\frac{(p+1)}{p}\right)\right] R_2^{(2p+1)}} \tag{D.5}$$

Solving Equation (D.3a) and (D.3e) for A_{p1} and equating the results to solve for B_{p2}, yields

$$B_{p2} = \frac{\mu_1 J_{p1}(\theta)}{(2p+1) R_1^{-(p+2)} P_p^1(\cos\theta)} \equiv \left[Y\right] \tag{D.6}$$

Solving Equation (D.3c) and (D.3g) for B_{p4} and equating the results to solve for A_{p3}, yields

$$A_{p3} = B_{p3}\left[W\right] + \left[X\right]\left[S\right] \equiv \left[Z\right] \tag{D.7}$$

$$\left[W\right] = \frac{\left(\frac{\mu_1}{\mu_2} - 1\right)}{R_3^{(2p+1)}\left[1+ \frac{\mu_1}{\mu_2}\left(\frac{p+1}{p}\right)\right]} , \quad \left[S\right] = \frac{(2p+1)}{p\left[1+ \frac{\mu_1}{\mu_2}\left(\frac{p+1}{p}\right)\right]} \tag{D.8}$$

$$\left[X\right] = A_{p4} \tag{D.9}$$

Now, using Equation (D.3b) and the results of Equations (D.4) through (D.9) to solve for B_{p3}, yields

$$B_{p3} = \frac{R_2^p\left[X\right]\left[S\right] - R_2^p\left[X\right]\left[S\right]\left[T\right] - \left[Y\right]\left[A\right] R_2^p - \left[Y\right]R_2^{-(p+1)}}{R_2^p\left[W\right]\left[T\right] - R_2^{-(p+1)} - R_2^p\left[W\right]} \tag{D.10}$$

where

$$\left[X\right] = \frac{\mu_1 J_{p2}(\theta)}{(2p+1)\left(R_4^{(p-1)}\right) P_p^1(\cos\theta)} \tag{D.11}$$

$$\left[Y\right] = \frac{\mu_1 J_{p1}(\theta)}{(2p+1)\left(R_1^{-(p+2)}\right) P_p^1(\cos\theta)} \tag{D.12}$$

$$\left[T\right] = \frac{\left(\frac{(2p+1)}{p}\right) R_2^{(2p+1)}}{\left[\left(1 + \frac{\mu_2}{\mu_1}\left(\frac{(p+1)}{p}\right)\right)R_2^{(2p+1)}\right]} \tag{D.13}$$

$$[A] = \frac{\left(\frac{\mu_2}{\mu_1} - 1\right)}{\left[1+ \frac{\mu_2}{\mu_1}\left(\frac{(p+1)}{p}\right)\right] R^{(2p+1)}} \qquad \text{(D.14)}$$

$$[W] = \frac{\left(\frac{\mu_1}{\mu_2} - 1\right)}{R_3^{(2p+1)}\left[1 + \frac{\mu_1}{\mu_2}\left(\frac{(p+1)}{p}\right)\right]} \qquad \text{(D.15)}$$

Now B_{p4}, B_{p5}, and A_{p1} can be found from

$$B_{p4} = A_{p3}R_3^{(2p+1)} + B_{p3} - A_{p4}R_3^{(2p+1)} \qquad \text{(D.16)}$$

$$B_{p5} = - A_{p4}\left(\frac{(p+1)}{p}\right) R_4^{(2p+1)} + B_{p4} + \frac{\mu_1 J_{p2}(\theta)}{P_p^1(\cos\theta)\, pR_4^{-(p+2)}} \qquad \text{(D.17)}$$

$$A_{p1} = A_{p2} + B_{p2}R_1^{-(2p+1)} \qquad \text{(D.18)}$$

The constants have now been found. After the numerical values of B_{p3}, A_{p4}, and B_{p2} are calculated for a specific problem, the numerical values for the other coefficients can be obtained from the following equations

$$A_{p3} = B_{p3} \left[W\right] + \left[X\right]\left[S\right] \equiv \left[Z\right] \tag{D.19a}$$

$$B_{p4} = A_{p3}R_3^{(2p+1)} + B_{p3} - A_{p4}R_3^{(2p+1)} \tag{D.19b}$$

$$B_{p5} = -A_{p4}\left(\frac{(p+1)}{p}\right)R_4^{(2p+1)} + B_{p4}$$

$$+ \frac{\mu_1 J_{p2}(\theta)}{P_p^1(\cos\theta)\, pR_1^{-(p+2)}} \tag{D.19c}$$

$$A_{p2} = \left[T\right] A_{p3} + \left[A\right] B_{p2} \tag{D.19d}$$

$$A_{p1} = A_{p2} + B_{p2}R_1^{-(2p+1)} \tag{D.19e}$$

REDUCTION OF THE MAGNETIC POTENTIAL WHEN μ_2 EQUALS μ_1 AND $J_1(\theta) = 0$

The coefficients A_{p1}, A_{p2}, A_{p3}, A_{p4}, B_{p2}, B_{p3}, B_{p4}, and B_{p5} for the potentials are now evaluated for the system consisting of a ferromagnetic shell with permeability μ_2 surrounded by an infinitesimally thin current band (J_2) in a homogeneous medium with permeability μ_1, in the limit as $\mu_2 = \mu_1 (J_1) = 0)$. The variables are defined in Figure 9 located in the text. When μ_2 is set equal to μ_1 the problem reduces to that of finding the potentials in the two regions of a simple current band (see Appendix B of Reference 7), because the ferromagnetic shell will now have a permeability μ_1 equal to that of the homogeneous medium with permeability μ_1.

In this limit the coefficients should assume the following form:

$$A_{p1} = A_{p2} = A_{p3} = A_{p4} \tag{D.20a}$$

$$B_{p2} = B_{p3} = B_{p4} = 0 \tag{D.20b}$$

$$A_{p1} = B_{p5}\left(R_4^{-(2p+1)}\right) \tag{D.20c}$$

and where A_{p1} and B_{p5} should reduce to coefficients for the potentials in the two regions for the spherical band problem.[7] If the coefficients assume this mathematical form it will prove that the mathematical form of the coefficients for the spherical shell surrounded by a thin current band are mathematically correct.

From Equation (D.10) B_{p3} is

$$B_{p3} = \frac{R_2^p[X][S] - R_2^p[X][S][T] - [Y][A]\,R_2^p - [Y]\,R_2^{-(p+1)}}{R_2^p\,[W][T] - R_2^{-(p+1)} - R_2^p\,[W]} \tag{D.21}$$

Now

$$B_{p3}\Big|_{\mu_2=\mu_1} = \lim_{\mu_2=\mu_1} B_{p3} = 0 \tag{D.22}$$

because $$\left[Y\right] = 0 \quad \text{for } J_{p1} = 0$$

$$\underset{\mu_2=\mu_1}{\text{limit}} \left[T\right] = 1$$

The expression for B_{p2} is zero because $J_{p1}(\theta) = 0$. The expression for A_{p2} (see Equation (D.5)) is

$$A_{p2} = A_{p3} \left[T\right] + B_{p2} \left[A\right] \tag{D.23}$$

because $$\left[T\right] \Big|_{\mu_2=\mu_1} = 1 \text{ and } B_{p2} = 0$$

$$A_{p2} = A_{p3} \tag{D.24}$$

The expression for A_{p1} (see Equation (D.18)) is

$$A_{p1} = A_{p2} + B_{p2} R_1^{-(2p+1)} \tag{D.25}$$

and because $B_{p2} = 0$

$$A_{p1} = A_{p2} \tag{D.26}$$

The expression for A_{p3} (see Equation (D.7)) is

$$A_{p3} = B_{p3}\left[W\right] + \left[X\right]\left[S\right] \tag{D.27}$$

and because

$$B_{p3}\Big|_{\mu_2=\mu_1} = 0$$

$$A_{p3} = \left[X\right] \equiv A_{p4} \tag{D.28}$$

Now

$$A_{p1} = A_{p2} = A_{p3} = A_{p4} = \frac{\mu_1 J_{p2}(\theta)}{(2p+1)R_4^{(p-1)}P_p^1(\cos\theta)} \tag{D.29}$$

The expression for B_{p4} (see Equation (D.16)) is

$$B_{p4} = A_{p3}R_3^{(2p+1)} + B_{p3} - A_{p4}R_3^{(2p+1)} \tag{D.30}$$

Because in the limit $\mu_2 = \mu_1$, $A_{p3} = A_{p4}$, and because $B_{p3} = 0$ we have

$$B_{p4} = 0 \tag{D.31}$$

The expression for B_{p5} (see Equation (D.17)) is

$$B_{p5} = - A_{p4} \left(\frac{(p+1)}{p}\right) R_4^{(2p+1)} + B_{p4} + \frac{\mu_1 J_{p2}(\theta)}{pR_4^{-(p+2)} P_p^1(\cos\theta)} \tag{D.32}$$

In the limit

$$B_{p5}\Big|_{\mu_2=\mu_1} = \underset{\mu_2 \to \mu_1}{\ell imit}\ B_{p5} = \frac{\mu_1 J_{p2}(\theta)}{R_4^{-(p+2)}(2p+1) P_p^1(\cos\theta)} \tag{D.33}$$

This means that in the five regions, the potential used in Equations (D.1) reduce, when $\mu_2 = \mu_1$ and $J_{p1} = 0$, to the form

$$A_{\psi I,\ II,\ III,\ IV} = \sum_{p=1}^{\infty} \left(A_{p1}\Big|_{\mu_2=\mu_1} r^p \right) P_p^1(\cos\theta) \tag{D.34a}$$

$$A_{\psi V} = \sum_{p=1}^{\infty} \left(B_{p5}\Big|_{\mu_2=\mu_1} \frac{1}{r^{p+1}} \right) P_p^1(\cos\theta) \tag{D.34b}$$

because

$$A_{p1} = A_{p2} = A_{p3} = A_{p4} = \frac{\mu_1 J_{p2}(\theta)}{(2p+1) R_4^{(p-1)} P_p^1(\cos\theta)} \tag{D.35a}$$

$$B_{p2} = B_{p3} = B_{p4} = 0 \tag{D.35b}$$

$$B_{p5} = \frac{\mu_1 J_{p2}(\theta)}{R_4^{-(p+2)}(2p+1)P_p^1(\cos\theta)} \tag{D.35c}$$

For comparison, the coefficients in Reference 7 are (primes are used for distinction)

$$A'_{p1} = B'_{p2} R_1^{(-2p-1)} \tag{D.36a}$$

$$B'_{p2} = \frac{\mu_1 J_p(\theta)}{R_1^{-(p+2)}(2p+1)P_p^1(\cos\theta)} \tag{D.36b}$$

When making the comparison one lets $R_4 = R_1$.

REDUCTION OF THE MAGNETIC POTENTIAL WHEN μ_2 EQUALS μ_1 AND $J_2(\theta) = 0$

In a manner similar to that of the preceeding section, the coefficients are evaluated for the system consisting of an infinitesimally thin current band $\left[J_1(\theta)\right]$ surrounded by a ferromagnetic shell with permeability μ_2, in a homogeneous medium with permeability μ_1, in the limit as $\mu_2 = \mu_1$.

In this limit the coefficients should assume the following form

$$A_{p2} = A_{p3} = A_{p4} = 0 \tag{D.37a}$$

$$A_{p1} = B_{p2} R_1^{-(2p+1)} \tag{D.37b}$$

$$B_{p2} = B_{p3} = B_{p4} = B_{p5} \tag{D.37c}$$

From Equation (D.5), A_{p2} is

$$A_{p2} = -\frac{A_{p3}\left(\frac{(2p+1)}{p}\right) R_2^{(2p+1)} + \left(\frac{\mu_2}{\mu_1} - 1\right) B_{p2}}{\left[1 + \frac{\mu_2}{\mu_1}\left(\frac{(p+1)}{p}\right)\right] R_2^{(2p+1)}} \tag{D.38}$$

$$A_{p2}\Big|_{\mu_2=\mu_1} = \underset{\mu_2=\mu_1}{\ell\text{imit}}\; A_{p2} = A_{p3} \tag{D.39}$$

The expression for B_{p2} (see Equation (D.6)) is

$$B_{p2} = \frac{\mu_1 J_{p1}(\theta)}{(2p+1) R_1^{-(p+2)} P_p^1(\cos\theta)} \equiv [Y] \tag{D.40}$$

The expression for A_{p4} (see Equation (D.4)) is

$$A_{p4} = \frac{\mu_1 J_{p2}(\theta)}{(2p+1) R_4^{(p-1)} P_p^1(\cos\theta)} \equiv [X] \tag{D.41}$$

and because $J_{p2}(\theta) = 0$

$$A_{p4} = 0 \tag{D.42}$$

The expression for B_{p3} (see Equation (D.10)) reduces when $\mu_2 = \mu_1$ to

$$B_{p3}\Big|_{\mu_2=\mu_1} = \frac{\mu_1 J_{p1}(\theta)}{(2p+1)R_1^{-(p+2)}P_p^1(\cos\theta)} \tag{D.43}$$

because, when $\mu_2 = \mu_1$ and $J_{p2} = 0$

$$\left[X\right] = 0 \tag{D.44a}$$

$$\left[Y\right] = \frac{\mu_1 J_{p1}(\theta)}{(2p+1)R_1^{-(p+2)}P_p^1(\cos\theta)} \tag{D.44b}$$

$$\left[T\right] = 1 \tag{D.44c}$$

$$\left[A\right] = 0 \tag{D.44d}$$

$$\left[W\right] = 0 \tag{D.44e}$$

The expression for A_{p3} (Equation (D.7)) is

$$A_{p3} = B_{p3}\left[W\right] + \left[X\right]\left[S\right] \tag{D.45}$$

and because $\left[W\right] = \left[X\right] = 0$ when $\mu_2 = \mu_1$, and $J_{p2} = 0$, we have

$$A_{p3} = 0 \tag{D.46}$$

Similarly, for A_{p2}

$$A_{p2} = A_{p3} \left[T\right] + B_{p2} \left[A\right] \tag{D.47}$$

and because $A_{p3} = 0$ and $\left[A\right] = 0$ when $\mu_2 = \mu_1$, we have

$$A_{p2} = 0 \tag{D.48}$$

The expression for A_{p1} (Equation (D.18)) is

$$A_{p1} = A_{p2} + B_{p2}R_1^{-(2p+1)} \tag{D.49}$$

Because $A_{p2} = 0$, Equation (D.49) yields

$$A_{p1} = B_{p2}R_1^{-(2p+1)} \tag{D.50}$$

For B_{p4}, using Equations (D.16), (D.46), and (D.42) we have

$$B_{p4} = B_{p3} \tag{D.51}$$

Similarly, for B_{p5}, using Equations (D.17) and (D.42), and the fact that $J_{p2} = 0$, we have

$$B_{p5} = B_{p4} \tag{D.52}$$

Thus, we have shown that when $\mu_2 = \mu_1$ and $J_{p2}(\theta) = 0$,

$$A_{p2} = A_{p3} = A_{p4} = 0 \tag{D.53a}$$

$$A_{p1} = B_{p2}R_1^{-(2p+1)} \tag{D.53b}$$

$$B_{p2} = B_{p3} = B_{p4} = B_{p5} = \frac{\mu_1 J_{p1}(\theta)}{(2p+1)R_1^{-(p+2)}P_p^1(\cos\theta)} \tag{D.53c}$$

Once again this means that, in the five regions, the potentials used in Equations (D.1) reduce when $\mu_1 = \mu_2$ and $J_{p2} = 0$ to the form

$$A_{\psi I} = \sum_{p=1}^{\infty} \left(A_{p1}\Big|_{\mu_2=\mu_1} r^p \right) P_p^1(\cos\theta) \tag{D.54a}$$

$$A_{\psi II,III,IV,V} = \sum_{p=1}^{\infty} \left(B_{p2}\Big|_{\mu_2=\mu_1} \frac{1}{r^{(p+1)}} \right) P_p^1(\cos\theta) \tag{D.54b}$$

For comparison, the coefficients in Reference 7 (primes are used for distinction) are

$$A'_{p1} = B'_{p2} R_1^{(-2p-1)} \tag{D.55a}$$

$$B'_{p2} = \frac{\mu_1 J_p(\theta)}{R_1^{-(p+2)}(2p+1)P_p^1(\cos\theta)} \tag{D.55b}$$

APPENDIX E

FERROMAGNETIC PROLATE SPHEROIDAL BODIES IN A CONSTANT EXTERNAL INDUCING FIELD

INTRODUCTION

In previous work Brown and Baker[8,9] derived the closed form mathematical expressions for the magnetic flux density for various configurations of a ferromagnetic spheroidal body surrounding and/or surrounded by a stationary current band of azimuthal symmetry. The problem of determining the magnetic induction for a prolate spheroidal body surrounding and/or surrounded by an infinitesimally thin current band can be generalized to include an external magnetic field. The superposition principle discussed in the text of this report can be used in these cases to include a constant external magnetic field. The magnetic induction in each region for a three-dimensional magnetic spheroidal shell in an arbitrary external magnetic field $\overline{H}_o$ is added to the magnetic induction for the corresponding region for the spheroidal shell surrounding and/or surrounded by a stationary current band. The problem of deriving the magnetic induction for a current band of finite width surrounding a solid ferromagnetic spheroid can also be generalized to include an external magnetic field $\overline{H}_o$ in a similar manner. Thus, the magnetic induction for a ferromagnetic spheroidal body in an external magnetic field must be determined.

Both constant external field problems were solved by Nixon[6] of the Center. The closed form mathematical solutions for the magnetic induction for both constant external field problems were presented in Reference 6 in Cartesian coordinates. It was necessary to convert these mathematical expressions to spheroidal coordinates to be compatible with this work.

SOLID FERROMAGNETIC SPHEROID IN AN EXTERNAL INDUCING FIELD

The solid ferromagnetic prolate spheroid in a constant external inducing field is shown in Figure (E.1). The permeability of the solid spheroid is μ_2 and the boundary of the is determined by $\eta = \eta_1 =$ constant. The permeability μ_o of vacuum that is external to the spheroid is denoted by μ_1. The constant arbitrary magnetic field is designated as $\overline{H}_o$.

It is assumed that μ_2 in the spheroid is constant, and that μ_1 is constant in the region external to the spheroid. Because there are no currents in any regions in the problem, the magnetic field $\overline{H}$ can be expressed as the negative of the

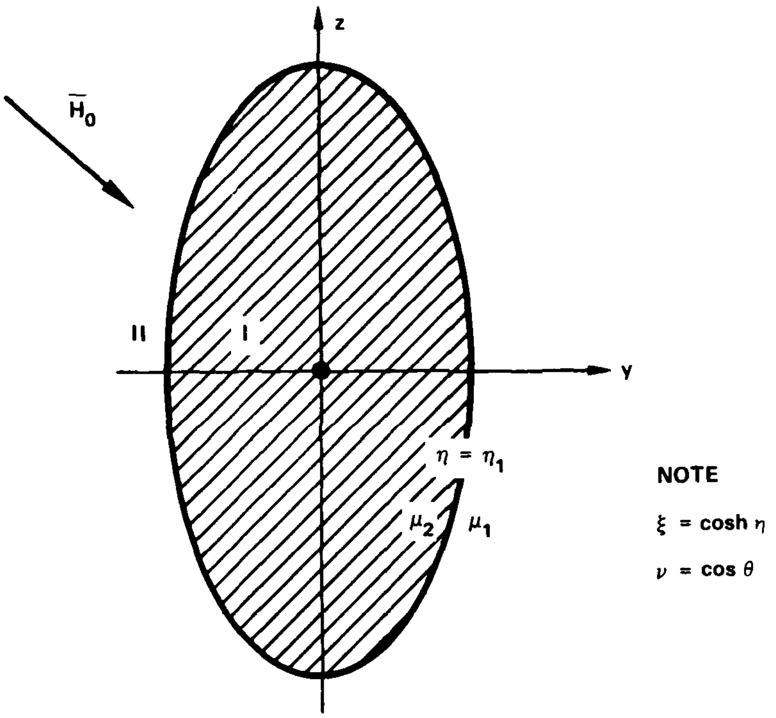


Figure E.1 - Ferromagnetic Prolate Spheroidal Solid in a Constant External Magnetic Field gradient of a magnetic scalar potential ϕ_m in regions I and II, respectively.

$$\overline{H}_I = - \overline{\nabla}\phi_{Im} \qquad \text{for } 0 \leq \eta \leq \eta_1 \tag{E.1a}$$

$$\overline{H}_{II} = - \overline{\nabla}\phi_{II} \qquad \text{for } \eta_1 \leq \eta < \infty \tag{E.1b}$$

where

$$\overline{B}_1 = \mu_2 \overline{H}_1 \tag{E.1c}$$

$$\overline{B}_{II} = \mu_1 \overline{H}_I \tag{E.1d}$$

The major step toward solving this problem is to determine the solutions of the scalar Laplace's equation in regions I and II which satisfy the boundary conditions at $\eta = \eta_1$. In terms of $\overline{B}$ and $\overline{H}$, the magnetostatic boundary conditions are

$$(\overline{B}_{II} - \overline{B}_I) \cdot \overline{n}_{12} = 0 \qquad \text{at } \eta = \eta_1 \tag{E.2a}$$

$$\overline{n}_{12} \times (\overline{H}_{II} - \overline{H}_I) = 0 \qquad \text{at } \eta = \eta_1 \tag{E.2b}$$

where $\overline{n}_{12}$ is the unit vector normal to the surface of the spheroid, outward from region I to region II. The general expression of ϕ_{Im} and ϕ_{IIm} which satisfy Laplace's equation in regions I and II, are:

$$\phi_{Im} = A\xi\nu + B\left[(\xi^2-1)(1-\nu^2)\right]^{\frac{1}{2}}\cos\psi + M\left[(\xi^2-1)(1-\nu^2)\right]^{\frac{1}{2}}\sin\psi \tag{E.3a}$$

$$\phi_{IIm} = D\left[-2+\xi\ell n\left(\frac{\xi+1}{\xi-1}\right)\right]\nu + (E\cos\psi+F\sin\psi)\left[\frac{2\xi}{\left(\xi^2-1\right)^{\frac{1}{2}}} - (\xi^2-1)^{\frac{1}{2}}\ell n\left(\frac{\xi+1}{\xi-1}\right)\right](1-\nu^2)^{\frac{1}{2}}$$

$$- H_{oz}a\xi\nu - (H_{ox}\cos\psi + H_{oy}\sin\psi)\, a\left[(\xi^2-1)(1-\nu^2)\right]^{\frac{1}{2}} \tag{E.3b}$$

where a is one-half of the focal length.

The coefficients are determined by the magnetostatic boundary conditions, Equations (E.2a) and (E.2b).

$$A = \frac{2\mu_1\ a\ H_{oz}}{D_1}\left[\frac{-\xi_1^2}{\xi_1^2-1} + 1\right] \tag{E.4a}$$

$$D = \frac{H_{oz}\ a\ \xi_1}{D_1}\ (\mu_2-\mu_1) \tag{E.4b}$$

$$B = \frac{-\ 4\mu_1\ H_{ox}\ a}{\left(\xi_1^2-1\right)D_2} \tag{E.4c}$$

$$E = \frac{H_{ox}\ a\ \xi_1(\mu_2-\mu_1)}{D_2} \tag{E.4d}$$

$$M = \frac{-4\mu_1 H_{oy} a}{\left(\xi_1^2-1\right) D_2} \tag{E.4e}$$

$$F = \frac{H_{oy} a \xi_1 (\mu_2-\mu_1)}{D_2} \tag{E.4f}$$

where

$$\xi_1 = \cosh \eta_1 \tag{E.4g}$$

$$D_1 = \frac{2\mu_1 \xi_1^2}{\left(\xi_1^2-1\right)} - 2\mu_2 + \xi_1(\mu_2-\mu_1)\ln\left(\frac{\xi_1+1}{\xi_1-1}\right) \tag{E.4h}$$

$$D_2 = \frac{2}{\xi_1^2-1}\left(\mu_1+\mu_2\xi_1^2\right) - 2\mu_1 + \xi_1(\mu_1-\mu_2)\ln\left(\frac{\xi_1+1}{\xi_1-1}\right) \tag{E.4i}$$

For details of this derivation the reader should consult Nixon.[6] The authors have changed his mathematical expressions for the magnetic induction in regions I and II from Cartesian to spheroidal coordinates. This makes the expressions for the magnetic induction compatible with the work presented in the text of this report. These mathematical expressions are, in regions I and II,

$$B_{\eta I} = \frac{-\mu_2(\sinh\eta)}{a(\sinh^2\eta + \sin^2\theta)^{\frac{1}{2}}}\left\{\left[A\cos\theta + \frac{B\cosh\eta(1-\cos^2\theta)^{\frac{1}{2}}}{(\cosh^2\eta-1)^{\frac{1}{2}}}\cos\psi\right.\right.$$

$$\left.\left. + \frac{M\cosh\eta(1-\cos^2\theta)^{\frac{1}{2}}}{(\cosh^2\eta-1)^{\frac{1}{2}}}\sin\psi\right]\right\} \tag{E.5a}$$

$$B_{\theta I} = \frac{\mu_2(\sin\theta)}{a(\sinh^2\eta + \sin^2\theta)^{\frac{1}{2}}}\left\{\left[A\cosh\eta - (B\cos\psi + M\sin\psi)\frac{(\cosh^2\eta-1)^{\frac{1}{2}}}{(1-\cos^2\theta)^{\frac{1}{2}}}\cos\theta\right]\right\}$$

$$B_{\psi I} = \frac{-\mu_2}{a(\sinh\eta\sin\theta)}\left\{(-B\sin\psi + M\cos\psi)(\cosh^2\eta-1)^{\frac{1}{2}}(1-\cos^2\theta)^{\frac{1}{2}}\right\} \tag{E.5b}$$

$$B_{\eta II} = \frac{-\mu_1(\sinh\eta)}{a(\sinh^2\eta + \sin^2\theta)^{\frac{1}{2}}}\left\{D\left[-\frac{2\cosh\eta}{\cosh^2\eta-1} + \ln\left(\frac{\cosh\eta+1}{\cosh\eta-1}\right)\right]\cos\theta\right.$$

$$+\left[-\frac{2}{(\cosh^2\eta-1)^{3/2}} + \frac{2}{(\cosh^2\eta-1)^{\frac{1}{2}}} - \frac{\cosh\eta}{(\cosh^2\eta-1)^{\frac{1}{2}}}\ln\left(\frac{\cosh\eta+1}{\cosh\eta-1}\right)\right]$$

$$\left[(1-\cos^2\theta)^{\frac{1}{2}}(E\cos\psi + F\sin\psi)\right] - H_{oz}\,a\cos\theta$$

$$\left. - a\left(H_{ox}\cos\psi + H_{oy}\sin\psi\right)\cosh\eta\,\frac{(1-\cos^2\theta)^{\frac{1}{2}}}{(\cosh^2\eta-1)^{\frac{1}{2}}}\right\} \tag{E.5c}$$

$$
B_{\theta II} = \frac{\mu_1(\sin\theta)}{a\left(\sinh^2\eta + \sin^2\theta\right)^{\frac{1}{2}}} \left\{ D\left[-2 + \cosh\eta \,\ell n\left(\frac{\cosh\eta+1}{\cosh\eta-1}\right)\right] \right.
$$

$$
- \left[\left(E\cos\psi + F\sin\psi\right) \frac{\cos\theta}{\left(1-\cos^2\theta\right)^{\frac{1}{2}}}\right]\left[\frac{2\cosh\eta}{\left(\cosh^2\eta-1\right)^{\frac{1}{2}}}\right.
$$

$$
\left. - \left(\cosh^2\eta-1\right)^{\frac{1}{2}} \ell n\left(\frac{\cosh\eta+1}{\cosh\eta-1}\right)\right] - H_{oz}\, a\cosh\eta
$$

$$
\left. + \left(H_{ox}\cos\psi + H_{oy}\sin\psi\right) a \frac{\cos\theta\left(\cosh^2\eta-1\right)^{\frac{1}{2}}}{\left(1-\cos^2\theta\right)^{\frac{1}{2}}} \right\} \tag{E.5d}
$$

$$
B_{\psi II} = \frac{-\mu_1}{a(\sinh\eta\,\sin\theta)} \left(\left\{\left(-E\sin\psi + F\cos\psi\right)\left(1-\cos^2\theta\right)^{\frac{1}{2}}\right.\right.
$$

$$
\left.\left[\frac{2\cosh\eta}{\left(\cosh^2\eta-1\right)^{\frac{1}{2}}} - \left(\cosh^2\eta-1\right)^{\frac{1}{2}} \ell n\left(\frac{\cosh\eta+1}{\cosh\eta-1}\right)\right]\right\}
$$

$$
\left. + \left(H_{ox}\sin\psi - H_{oy}\cos\psi\right) a \left(\cosh^2\eta-1\right)^{\frac{1}{2}} \left(1-\cos^2\theta\right)^{\frac{1}{2}} \right) \tag{E.5e}
$$

FERROMAGNETIC PROLATE SPHEROIDAL SHELL IN AN EXTERNAL INDUCING FIELD

The problem of the spheroidal shell is similar to the problem of the solid spheroid. The inner boundary of the prolate spheroidal is η_1 and the outer boundary is η_2 (see Figure E.2)). The permeability of the magnetic material in the shell is μ_2 and the permeability μ_o of vacuum that is internal and external to the shell is denoted by μ_1. The constant external magnetic field is designated by $\overline{H}_o$.

The problem of deriving the closed form mathematical expressions for the magnetic flux density in each of the three regions (I through III) was solved in detail by Nixon.[6] The problem was solved in a method exactly analogous to the method used to solve the problem of a solid spheroid in a constant external magnetic field. For details of the derivation, Reference 6 should be consulted.

The general expressions for the magnetic scalar potential ϕ_m in regions I through III are

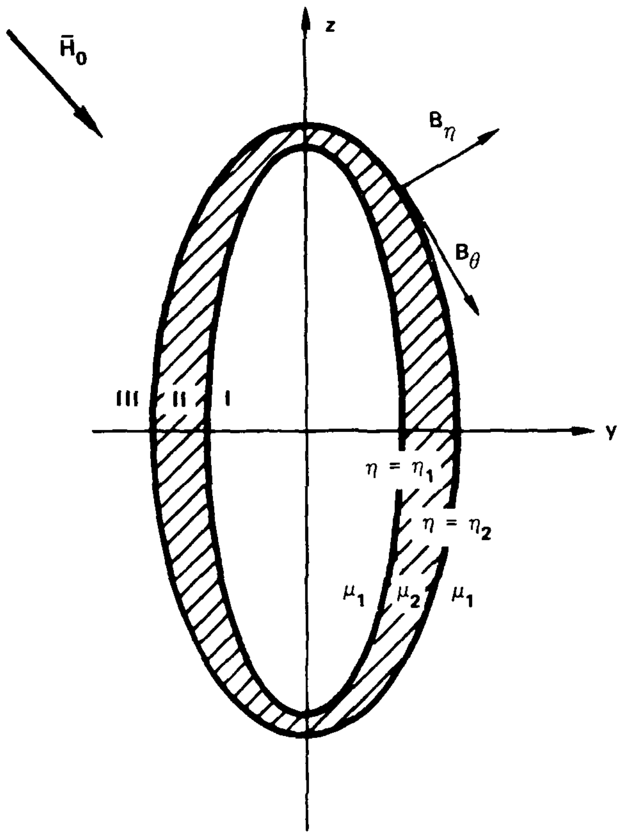
$$\phi_{Im} = A' \ \xi \ \nu + B' \left[\left(\xi^2-1\right)\left(1-\nu^2\right)\right]^{\frac{1}{2}} \cos \psi$$

$$+ M' \left[\left(\xi^2-1\right)\left(1-\nu^2\right)\right]^{\frac{1}{2}} \sin \psi \qquad \text{(E.6a)}$$

$$\phi_{IIm} = \left[G'\xi + H'\left(-2 + \xi \ \ell n \ \left(\frac{\xi+1}{\xi-1}\right)\right)\right]\nu + \left[I'\left(\xi^2-1\right)^{\frac{1}{2}}\right.$$

$$\left. + J'\left(\frac{2\xi}{\left(\xi^2-1\right)^{\frac{1}{2}}} - \left(\xi^2-1\right)^{\frac{1}{2}} \ell n \left(\frac{\xi+1}{\xi-1}\right)\right)\right]\left(1-\nu^2\right)^{\frac{1}{2}} \cos \psi$$

$$+ \left[K'\left(\xi^2-1\right)^{\frac{1}{2}} + L'\left(\frac{2\xi}{\left(\xi^2-1\right)^{\frac{1}{2}}} - \left(\xi^2-1\right)^{\frac{1}{2}} \ell n \left(\frac{\xi+1}{\xi-1}\right)\right)\right]\left(1-\nu^2\right)^{\frac{1}{2}} \sin \psi \qquad \text{(E.6b)}$$



NOTE

ξ = COSH η

ν = COS θ

Figure E.2 - Ferromagnetic Prolate Spheroidal Shell in a Constant External Magnetic Field

$$\phi_{IIIm} = D\left[-2 + \xi \ln\left(\frac{\xi+1}{\xi-1}\right)\right]\nu + \left[E' \cos\psi + F' \sin\psi\right]$$

$$\left[\frac{2\xi}{\left(\xi^2-1\right)^{\frac{1}{2}}} - \left(\xi^2-1\right)^{\frac{1}{2}} \ln\left(\frac{\xi+1}{\xi-1}\right)\right]\left(1-\nu^2\right)^{\frac{1}{2}} - H_{oz}\, a\, \xi\, \nu$$

$$-\left[H_{ox} \cos\psi + H_{oy} \sin\psi\right] a \left[\left(\xi^2-1\right)^{\frac{1}{2}} \left(1-\nu^2\right)^{\frac{1}{2}}\right] \qquad \text{(E.6c)}$$

The coefficients are determined by the usual magnetostatic boundary conditions on the spheroidal surfaces at $\eta = \eta_1$ and $\eta = \eta_2$. The coefficients determined by this method are

$$A' = \frac{H_{oz}\, a\, \mu_2\mu_1}{D_1}\left(2 - \frac{2\xi_1^{\,2}}{\xi_1^2-1}\right)\left(-2 + \frac{2\xi_2^{\,2}}{\xi_2^2-1}\right) \qquad \text{(E.7)}$$

$$G' = \frac{H_{oz}\, a\, \mu_1}{D_1}\left(-2 + \frac{2\xi_2^{\,2}}{\xi_2^{\,2}-1}\right)\left\{\mu_1\left[2 - \xi_1 \ln\left(\frac{\xi_1+1}{\xi_1-1}\right)\right]\right.$$

$$\left. - \mu_2\xi_1\left[\frac{2\xi_1}{\xi_1^{\,2}-1} - \ln\left(\frac{\xi_1+1}{\xi_1-1}\right)\right]\right\} \qquad \text{(E.8)}$$

$$D' = \frac{1}{D_1}\left[-H_{oz}\, a\, \xi_1\left\{\xi_2\mu_2^2\left[\frac{2\xi_2}{\xi_2^2-1} - \ln\left(\frac{\xi_2+1}{\xi_2-1}\right) - \frac{2\xi_1}{\xi_1^2-1} + \ln\left(\frac{\xi_1+1}{\xi_1-1}\right)\right]\right.\right. \qquad \text{(E.9)}$$

$$- \mu_2\mu_1 \left[2 - \xi_2 \ln\left(\frac{\xi_2+1}{\xi_2-1}\right) - \frac{2\xi_2\xi_1}{\xi_1^2-1} + \xi_2 \ln\left(\frac{\xi_1+1}{\xi_1-1}\right)\right]\Bigg\}$$

$$- H_{oz}\, a\, \mu_1 \Bigg\{ \xi_2\mu_2 \left[2 - \xi_1 \ln\left(\frac{\xi_1+1}{\xi_1-1}\right) - \frac{2\xi_2\xi_1}{\xi^2-1} + \xi_1 \ln\left(\frac{\xi_2+1}{\xi_2-1}\right)\right]$$

$$+ \mu_1 \left(\left[\ln\left(\frac{\xi_1+1}{\xi_1-1}\right) - \ln\left(\frac{\xi_2+1}{\xi_2-1}\right)\right] \xi_2\xi_1 - 2\left(\xi_2-\xi_1\right)\right)\Bigg\}\Bigg]$$

$$H' = \frac{H_{oz}\, a\, \mu_1\xi_1}{D_1}\left(-2 + \frac{2\xi_2^2}{\xi_2^2-1}\right)\left(\mu_1-\mu_2\right) \tag{E.10}$$

$$D_1 = -\xi_1 \Bigg\{\left[-2 + \xi_2 \ln\left(\frac{\xi_2+1}{\xi_2-1}\right)\right]\left[\frac{2\xi_2}{\xi_2^2-1} - \ln\left(\frac{\xi_2+1}{\xi_2-1}\right) - \frac{2\xi_1}{\xi_1^2-1} + \ln\left(\frac{\xi_1+1}{\xi_1-1}\right)\right]\mu_2^2$$

$$- \mu_1 \left[\ln\left(\frac{\xi_2+1}{\xi_2-1}\right) - \frac{2\xi_2}{\xi_2^2-1}\right]\left[2 - \xi_2 \ln\left(\frac{\xi_2+1}{\xi_2-1}\right) - \frac{2\xi_2\xi_1}{\xi_1^2-1}\right.$$

$$\left. + \xi_2 \ln\left(\frac{\xi_1+1}{\xi_1-1}\right)\right]\mu_2\Bigg\} - \mu_1 \Bigg\{\left[-2 + \xi_2 \ln\left(\frac{\xi_2+1}{\xi_2-1}\right)\right]\left[2 - \xi_1 \ln\left(\frac{\xi_1+1}{\xi_1-1}\right)\right.$$

$$\left. - \frac{2\xi_2\xi_1}{\xi_2^2-1} + \xi_1 \ln\left(\frac{\xi_2+1}{\xi_2-1}\right)\right]\mu_2 + \mu_1 \left[\ln\left(\frac{\xi_2+1}{\xi_2-1}\right) - \frac{2\xi_2}{\xi_2^2-1}\right]$$

(Note: Above equation continued on next page).

$$\cdot \left[\left(\ln\left(\frac{\xi_1+1}{\xi_1-1}\right) - \ln\left(\frac{\xi_2+1}{\xi_2-1}\right)\right)\xi_2\xi_1 - 2\left(\xi_2-\xi_1\right)\right]\Bigg\} \tag{E.11}$$

$$B' = \frac{H_{ox}\ a\ \mu_2\mu_1}{D_2}\Bigg\{-\left(\xi_2^2-1\right)^{\frac{1}{2}}\left[-\frac{2}{\left(\xi_2^2-1\right)^{3/2}} + \frac{2}{\left(\xi_2^2-1\right)^{\frac{1}{2}}} - \frac{\xi_2}{\left(\xi_2^2-1\right)^{\frac{1}{2}}}\ln\left(\frac{\xi_2+1}{\xi_1-1}\right)\right]$$

$$+ \frac{\xi_2}{\left(\xi_2^2-1\right)^{\frac{1}{2}}}\left[\frac{2\xi_2}{\left(\xi_2^2-1\right)^{\frac{1}{2}}} - \left(\xi_2^2-1\right)^{\frac{1}{2}}\ln\left(\frac{\xi_2+1}{\xi_2-1}\right)\right]\Bigg\}$$

$$\Bigg\{\left(\xi_1^2-1\right)^{\frac{1}{2}}\left[-\frac{2}{\left(\xi_1^2-1\right)^{3/2}} + \frac{2}{\left(\xi_1^2-1\right)^{\frac{1}{2}}} - \frac{\xi_1}{\left(\xi_1^2-1\right)^{\frac{1}{2}}}\ln\left(\frac{\xi_{1+1}}{\xi_{1-1}}\right)\right]$$

$$- \frac{\xi_1}{\left(\xi_1^2-1\right)^{\frac{1}{2}}}\left[\frac{2\xi_1}{\left(\xi_1^2-1\right)^{\frac{1}{2}}} - \left(\xi_1^2-1\right)^{\frac{1}{2}}\ln\left(\frac{\xi_1+1}{\xi_1-1}\right)\right]\Bigg\} \tag{E.12}$$

$$E' = \frac{1}{D_2}\Bigg\{H_{ox}\ a\ \mu_2\left(\xi_1^2-1\right)^{\frac{1}{2}}\left(\frac{\xi_2\mu_1}{\left(\xi_2^2-1\right)^{\frac{1}{2}}}\Bigg\{\left(\xi_2^2-1\right)^{\frac{1}{2}}\left[-\frac{2}{\left(\xi_1^2-1\right)^{3/2}} + \frac{2}{\left(\xi_1^2-1\right)^{\frac{1}{2}}}\right.\right.$$

$$\left.- \frac{\xi_1}{\left(\xi_1^2-1\right)^{\frac{1}{2}}}\ln\left(\frac{\xi_1+1}{\xi_1-1}\right)\right] - \frac{\xi_1}{\left(\xi_1^2-1\right)^{\frac{1}{2}}}\left[\frac{2\xi_2}{\left(\xi_2^2-1\right)^{\frac{1}{2}}} - \left(\xi_2^2-1\right)^{\frac{1}{2}}\ln\left(\frac{\xi_2+1}{\xi_2-1}\right)\right]\Bigg\}$$

(Note: Above equation continued on next pa :).

$$
\begin{aligned}
&- \mu_2 \left(\xi_2^2-1\right)^{\frac{1}{2}} \Bigg\{ \frac{\xi_2}{\left(\xi_2^2-1\right)^{\frac{1}{2}}} \left[- \frac{2}{\left(\xi_1^2-1\right)^{3/2}} + \frac{2}{\left(\xi_1^2-1\right)^{\frac{1}{2}}} - \frac{\xi_1}{\left(\xi_1^2-1\right)^{\frac{1}{2}}} \ln\left(\frac{\xi_1+1}{\xi_1-1}\right)\right] \\
&- \frac{\xi_1}{\left(\xi_1^2-1\right)^{\frac{1}{2}}} \left[- \frac{2}{\left(\xi_2^2-1\right)^{3/2}} + \frac{2}{\left(\xi_2^2-1\right)^{\frac{1}{2}}} - \frac{\xi_2}{\left(\xi_2^2-1\right)^{\frac{1}{2}}} \ln\left(\frac{\xi_2+1}{\xi_2-1}\right)\right] \Bigg\} \Bigg) \\
&\frac{- H_{ox}\, a\, \mu_1 \xi_1}{\left(\xi_1^2-1\right)^{\frac{1}{2}}} \Bigg(\frac{\mu_1 \xi_2}{\left(\xi_2^2-1\right)^{\frac{1}{2}}} \Bigg\{ \left(\xi_2^2-1\right)^{\frac{1}{2}} \left[\frac{2\xi_1}{\left(\xi_1^2-1\right)^{\frac{1}{2}}} - \left(\xi_1^2-1\right)^{\frac{1}{2}} \ln\left(\frac{\xi_1+1}{\xi_1-1}\right)\right] \\
&- \left(\xi_1^2-1\right)^{\frac{1}{2}} \left[\frac{2\xi_2}{\left(\xi_2^2-1\right)^{\frac{1}{2}}} - \left(\xi_1^2-1\right) \ln\left(\frac{\xi_2+1}{\xi_2-1}\right)\right] \Bigg\} - \mu_2 \left(\xi_2^2-1\right)^{\frac{1}{2}} \Bigg\{ \left(\xi_1^2-1\right)^{\frac{1}{2}} \Bigg[- \frac{2}{\left(\xi_2^2-1\right)^{3/2}} \\
&+ \frac{2}{\left(\xi_2^2-1\right)^{\frac{1}{2}}} - \frac{\xi_2}{\left(\xi_2^2-1\right)^{\frac{1}{2}}} \ln\left(\frac{\xi_2+1}{\xi_2-1}\right) \Bigg] - \frac{\xi_2}{\left(\xi_2^2-1\right)^{\frac{1}{2}}} \left[\frac{2\xi_2}{\left(\xi_2^2-1\right)^{\frac{1}{2}}} - (\xi_2^2-1)^{\frac{1}{2}} \ln\left(\frac{\xi_2+1}{\xi_2-1}\right)\right] \Bigg\} \Bigg) \Bigg\} .
\end{aligned}
\tag{E.13}
$$

$$
\begin{aligned}
I' = \frac{H_{ox}\, a\, \mu_1}{D_2} \Bigg\{ & \frac{\mu_2\, \xi_2 \left(\xi_1^2-1\right)^{1/2}}{\left(\xi_2^2-1\right)^{1/2}} \left[\frac{2\xi_2}{\left(\xi_2^2-1\right)^{1/2}} - \left(\xi_2^2-1\right)^{1/2} \ln\left(\frac{\xi_2+1}{\xi_2-1}\right)\right]\left[- \frac{2}{\left(\xi_1^2-1\right)^{3/2}} \right. \\
& \left. + \frac{2}{\left(\xi_1^2-1\right)^{1/2}} - \frac{\xi_1}{\left(\xi_1^2-1\right)^{1/2}} \ln\left(\frac{\xi_1+1}{\xi_1-1}\right)\right] - \mu_2 \left(\xi_2^2-1\right)^{1/2} \left(\xi_1^2-1\right)^{1/2} \left[- \frac{2}{\left(\xi_2^2-1\right)^{3/2}} \right. \\
& \left. + \frac{2}{\left(\xi_2^2-1\right)^{1/2}} - \frac{\xi_2}{\left(\xi_2^2-1\right)^{1/2}} \ln\left(\frac{\xi_2+1}{\xi_2-1}\right)\right]\left[- \frac{2}{\left(\xi_1^2-1\right)^{3/2}} - \frac{2}{\left(\xi_1^2-1\right)^{1/2}} \right. \\
& \left. + \frac{\xi_1}{\left(\xi_1^2-1\right)^{1/2}} \ln\left(\frac{\xi_1+1}{\xi_1-1}\right)\right] - \frac{\mu_1\, \xi_2 \xi_1}{\left(\xi_2^2-1\right)^{1/2} \left(\xi_1^2-1\right)^{1/2}} \left[\frac{2\xi_2}{\left(\xi_2^2-1\right)^{1/2}} - \left(\xi_2^2-1\right)^{1/2} \ln \right. \\
& \left. \left(\frac{\xi_2+1}{\xi_2-1}\right)\right]\left[\frac{2\xi_1}{\left(\xi_1^2-1\right)^{1/2}} - \left(\xi_1^2-1\right)^{1/2} \ln\left(\frac{\xi_1+1}{\xi_1-1}\right)\right] + \frac{\mu_1\, \xi_1 \left(\xi_1^2-1\right)^{1/2}}{\left(\xi_1^2-1\right)^{1/2}} \left[- \frac{2}{\left(\xi_1^2-1\right)^{3/2}} \right. \\
& \left. + \frac{2}{\left(\xi_2^2-1\right)^{1/2}} - \frac{\xi_2}{\left(\xi_2^2-1\right)^{1/2}} \ln\left(\frac{\xi_2+1}{\xi_2-1}\right)\right]\left[\frac{2\xi_1}{\left(\xi_1^2-1\right)^{1/2}} - \left(\xi_1^2-1\right)^{1/2} \ln\left(\frac{\xi_1+1}{\xi_1-1}\right)\right] \Bigg\}
\end{aligned}
\qquad \text{(E.14)}
$$

$$J' = \frac{H_{ox}\ a\ \mu_1}{D_2}\Bigg\{-\left[\frac{\mu_2\ \xi_2\xi_1}{\left(\xi_2^2-1\right)^{1/2}}\left(\frac{2\xi_2}{\left(\xi_2^2-1\right)^{1/2}}-\left(\xi_2^2-1\right)^{1/2}\ln\left(\frac{\xi_2+1}{\xi_2-1}\right)\right)\right]+\mu_2\xi_1\ \left(\xi_2^2-1\right)^{1/2}$$

$$\left[\frac{-2}{\left(\xi_2^2-1\right)^{3/2}}+\frac{2}{\left(\xi_2^2-1\right)^{1/2}}-\frac{\xi_2}{\left(\xi_2^2-1\right)^{1/2}}\ \ln\left(\frac{\xi_2+1}{\xi_2-1}\right)\right]$$

$$+\frac{\mu_1\xi_2\xi_1}{\left(\xi_2^2-1\right)^{1/2}}\left(\frac{2\xi_2}{\left(\xi_2^2-1\right)^{1/2}}-\left(\xi_2^2-1\right)^{1/2}\ln\left(\frac{\xi_2+1}{\xi_2-1}\right)\right)-\mu_1\xi_1\ \left(\xi_2^2-1\right)^{1/2}$$

$$\left[\frac{-2}{\left(\xi_2^2-1\right)^{3/2}}+\frac{2}{\left(\xi_2^2-1\right)^{1/2}}-\frac{\xi_2}{\left(\xi_2^2-1\right)^{1/2}}\ \ln\left(\frac{\xi_2+1}{\xi_2-1}\right)\right] \tag{E.15}$$

$$D_2 = -\left(\xi_1^2-1\right)^{1/2}\Bigg(\mu_2^2\left[\frac{2\xi_2}{\left(\xi_2^2-1\right)^{1/2}}-\left(\xi_2^2-1\right)^{1/2}\ln\left(\frac{\xi_2+1}{\xi_2-1}\right)\right]\Bigg\{\frac{\xi_2}{\left(\xi_2^2-1\right)^{1/2}}\Bigg[-\frac{2}{\left(\xi_1^2-1\right)^{3/2}}$$

$$+\frac{2}{\left(\xi_1^2-1\right)^{1/2}}-\frac{\xi_1}{\left(\xi_1^2-1\right)^{1/2}}\ \ln\left(\frac{\xi_1+1}{\xi_1-1}\right)\Bigg]-\frac{\xi_1}{\left(\xi_1^2-1\right)^{1/2}}\Bigg[-\frac{2}{\left(\xi_2^2-1\right)^{3/2}}+\frac{2}{\left(\xi_2^2-1\right)^{1/2}}$$

$$-\frac{\xi_2}{\left(\xi_2^2-1\right)^{1/2}}\ \ln\left(\frac{\xi_2+1}{\xi_2-1}\right)\Bigg]\Bigg\}-\mu_1\left[-\frac{2}{\left(\xi_2^2-1\right)^{3/2}}+\frac{2}{\left(\xi_2^2-1\right)^{1/2}}-\frac{\xi_2}{\left(\xi_2^2-1\right)^{1/2}}\ \ln\left(\frac{\xi_2+1}{\xi_2-1}\right)\right]$$

(Note: Above equation continued on next page).

$$
\Bigg\{\mu_2 \left(\xi_2^2-1\right)^{1/2} \Bigg[-\frac{2}{\left(\xi_1^2-1\right)^{3/2}} + \frac{2}{\left(\xi_1^2-1\right)^{1/2}} - \frac{\xi_1}{\left(\xi_1^2-1\right)^{1/2}} \ln\left(\frac{\xi_1+1}{\xi_1-1}\right)\Bigg]
$$

$$
- \frac{\mu_2\, \xi_1}{\left(\xi_1^2-1\right)^{1/2}} \Bigg[\frac{2\xi_2}{\left(\xi_2^2-1\right)^{1/2}} - \left(\xi_2^2-1\right)^{1/2} \ln\left(\frac{\xi_2+1}{\xi_2-1}\right)\Bigg]\Bigg\}\Bigg) - \frac{\mu_1\, \xi_1}{\left(\xi_1^2-1\right)^{1/2}} \Bigg(\mu_2 \Bigg[\frac{2\xi_2}{\left(\xi_2^2-1\right)^{1/2}}
$$

$$
- \left(\xi_2^2-1\right)^{1/2} \ln\left(\frac{\xi_2+1}{\xi_2-1}\right)\Bigg]\Bigg\{\left(\xi_1^2-1\right)^{1/2}\Bigg[-\frac{2}{\left(\xi_2^2-1\right)^{3/2}} + \frac{2}{\left(\xi_2^2-1\right)^{1/2}} - \frac{\xi_2}{\left(\xi_2^2-1\right)^{1/2}} \ln\left(\frac{\xi_2+1}{\xi_2-1}\right)\Bigg]
$$

$$
- \frac{\xi_2}{\left(\xi_2^2-1\right)^{1/2}}\Bigg[\frac{2\xi_1}{\left(\xi_1^2-1\right)^{1/2}} - \left(\xi_1^2-1\right)^{1/2} \ln\left(\frac{\xi_1+1}{\xi_1-1}\right)\Bigg]\Bigg\} + \mu_1\Bigg[-\frac{2}{\left(\xi_2^2-1\right)^{3/2}} + \frac{2}{\left(\xi_2^2-1\right)^{1/2}}
$$

$$
- \frac{\xi_2}{\left(\xi_2^2-1\right)^{1/2}} \ln\left(\frac{\xi_2+1}{\xi_2-1}\right)\Bigg]\Bigg\{\left(\xi_2^2-1\right)^{1/2}\Bigg[\frac{2\xi_1}{\left(\xi_1^2-1\right)^{1/2}} - \left(\xi_1^2-1\right)^{1/2} \ln\left(\frac{\xi_1+1}{\xi_1-1}\right)\Bigg]
$$

$$
- \left(\xi_1^2-1\right)^{1/2}\Bigg[\frac{2\xi_2}{\left(\xi_2^2-1\right)^{1/2}} - \left(\xi_2^2-1\right)^{1/2} \ln\left(\frac{\xi_2+1}{\xi_2-1}\right)\Bigg]\Bigg\}\Bigg) \tag{E.16}
$$

Using the symmetry conditions[6] that exist in this problem, the last four constants (M', F', K', and L') are obtained from the previous equations by simple substitution. Therefore, M', F', K', and L' are determined by substituting H_{oy} for H_{ox} in Equations (E.12) through (E.16).

The authors have changed Nixon's mathematical expressions for the magnetic induction in regions I through III from Cartesian to spheroidal coordinates. This makes the expressions for the magnetic induction compatible with the work presented in the text of this report. The mathematical expressions in regions I through III are

$$B_{\eta I} = \frac{-\mu_1 (\sinh \eta)}{a(\sinh^2\eta + \sin^2\theta)^{\frac{1}{2}}} \left\{ \left[A' \cos\theta + \frac{B' \cosh \eta (1-\cos^2\theta)^{\frac{1}{2}}}{(\cosh^2\eta - 1)^{\frac{1}{2}}} \cos\psi \right.\right.$$

$$\left.\left. + \frac{M' \cosh\eta \, (1-\cos^2\theta)^{\frac{1}{2}}}{(\cosh^2\eta-1)^{\frac{1}{2}}} \sin\psi \right] \right\} \qquad \text{(E.17a)}$$

$$B_{\theta I} = \frac{\mu_1 (\sin\theta)}{a(\sinh^2\eta + \sin^2\theta)^{\frac{1}{2}}} \left\{ \left[A' \cosh\eta - \left(B' \cos\psi + M' \sin\psi\right) \frac{(\cosh^2\eta-1)^{\frac{1}{2}}}{(1-\cos^2\theta)^{\frac{1}{2}}} \cos\theta \right] \right\} \qquad \text{(E.17b)}$$

$$B_{\psi I} = \frac{-\mu_1}{a \sinh\eta \sin\theta} \left\{ \left(-B' \sin\psi + M' \cos\psi\right) (\cosh^2\eta - 1)^{\frac{1}{2}} (1-\cos^2\theta)^{\frac{1}{2}} \right\} \qquad \text{(E.17c)}$$

$$B_{\eta II} = \frac{-\mu_2 (\sinh\eta)}{a(\sinh^2\eta + \sin^2\theta)^{\frac{1}{2}}} \left(G' + H' \left[-\frac{2\cosh\eta}{\cosh^2\eta - 1} + \ell n \left(\frac{\cosh\eta + 1}{\cosh\eta - 1} \right) \right] \right\} \cos\theta$$

(Note: Above equation is continued on next page.)

$$
+ \left(\frac{I' \cosh \eta}{(\cosh^2\eta - 1)^{\frac{1}{2}}} + J' \left[- \frac{2}{(\cosh^2\eta-1)^{3/2}} + \frac{2}{(\cosh^2\eta-1)^{\frac{1}{2}}} \right.\right.
$$

$$
\left.\left. - \frac{\cosh\eta}{(\cosh^2\eta-1)^{\frac{1}{2}}} \ln \left(\frac{\cosh\eta+1}{\cosh\eta-1} \right) \right] \right) (1-\cos^2\theta)^{\frac{1}{2}} \cos\psi + \left(\frac{K' \cosh\eta}{(\cosh^2\eta-1)^{\frac{1}{2}}} \right.
$$

$$
\left. + L' \left[- \frac{2}{(\cosh^2\eta-1)^{3/2}} + \frac{2}{(\cosh^2\eta-1)^{\frac{1}{2}}} - \frac{\cosh\eta}{(\cosh^2\eta-1)^{\frac{1}{2}}} \ln \left(\frac{\cosh\eta+1}{\cosh\eta-1} \right) \right] \right)
$$

$$
\left. (1 - \cos^2\theta)^{\frac{1}{2}} \sin\psi \right) \tag{E.18a}
$$

$$
B_{\theta II} = \frac{\mu_2 (\sin\theta)}{a(\sinh^2\eta + \sin^2\theta)^{\frac{1}{2}}} \left(\left\{ G' \cosh\eta + H' \left[-2 + \cosh\eta \ln \left(\frac{\cosh\eta+1}{\cosh\eta-1} \right) \right] \right\} \right.
$$

$$
- \left\{ I' (\cosh^2\eta-1)^{\frac{1}{2}} + J' \left[\frac{2\cosh\eta}{(\cosh^2\eta-1)^{\frac{1}{2}}} - (\cosh^2\eta-1)^{\frac{1}{2}} \ln \left(\frac{\cosh\eta+1}{\cosh\eta-1} \right) \right] \right\}
$$

$$
\frac{\cos\theta \cos\psi}{(1-\cos^2\theta)^{\frac{1}{2}}} - \left\{ K' (\cosh^2\eta-1)^{\frac{1}{2}} + L' \left[\frac{2\cosh\eta}{(\cosh^2\eta-1)^{\frac{1}{2}}} - (\cosh^2\eta-1)^{\frac{1}{2}} \right.\right.
$$

$$
\left.\left.\left. \ln \left(\frac{\cosh\eta+1}{\cosh\eta-1} \right) \right] \right\} \frac{\cos\theta \sin\psi}{(1-\cos^2\theta)^{\frac{1}{2}}} \right) \tag{E.18b}
$$

$$B_{II\psi} = \frac{\mu_2}{a \sinh \eta \sin \theta}\left(\left\{I' (\cosh^2\eta - 1)^{\frac{1}{2}} + J'\left[\frac{2 \cosh \eta}{(\cosh^2\eta-1)^{\frac{1}{2}}}\right.\right.\right.$$

$$\left.\left. - (\cosh^2\eta-1)^{\frac{1}{2}} \ln\left(\frac{\cosh\eta+1}{\cosh\eta-1}\right)\right]\right\}(1 - \cos^2\theta)^{\frac{1}{2}} \sin\psi + \left\{K'(\cosh^2\eta-1)^{\frac{1}{2}}\right.$$

$$\left.\left. - L'\left[\frac{2 \cosh \eta}{(\cosh^2\eta-1)^{\frac{1}{2}}} - (\cosh^2\eta-1)^{\frac{1}{2}} \ln\left(\frac{\cosh\eta+1}{\cosh\eta-1}\right)\right]\right\}(1-\cos^2\theta)^{\frac{1}{2}} \cos\psi\right) \tag{E.18c}$$

$$B_{\eta III} = \frac{-\mu_1(\sinh \eta)}{a(\sinh^2\eta + \sin^2\theta)^{\frac{1}{2}}}\left\{D'\left[\frac{-2 \cosh\eta}{\cosh^2\eta-1} + \ln\left(\frac{\cosh\eta+1}{\cosh\eta-1}\right)\right]\cos\theta\right.$$

$$+\left[\frac{-2}{(\cosh^2\eta-1)^{3/2}} + \frac{2}{(\cosh^2\eta-1)^{\frac{1}{2}}} - \frac{\cosh \eta}{(\cosh^2\eta-1)^{\frac{1}{2}}} \ln\left(\frac{\cosh\eta+1}{\cosh\eta-1}\right)\right]$$

$$\left[(1 - \cos^2\theta)^{\frac{1}{2}} (E'\cos\psi + F'\sin\psi)\right] - H_{oz}\, a \cos\theta$$

$$\left. - a\,(H_{ox} \cos\psi + H_{oy} \sin\psi) \cosh\eta \frac{(1-\cos^2\theta)^{\frac{1}{2}}}{(\cosh^2\eta-1)^{\frac{1}{2}}}\right\} \tag{E.19a}$$

$$
B_{\theta III} = \frac{\mu_1 (\sin\theta)}{a(\sinh^2\eta + \sin^2\theta)^{\frac{1}{2}}} \Bigg\{ D' \left[-2 + \cosh\eta \, \ln\left(\frac{\cosh\eta+1}{\cosh\eta-1}\right)\right]
$$

$$
- \left[(E' \cos\psi + F' \sin\psi) \left(\frac{\cos\theta}{(1-\cos^2\theta)^{\frac{1}{2}}}\right) \left[\frac{2\cosh\eta}{(\cosh^2\eta-1)^{\frac{1}{2}}}\right.\right.
$$

$$
\left.\left. - (\cosh^2\eta-1)^{\frac{1}{2}} \ln\left(\frac{\cosh\eta+1}{\cosh\eta-1}\right)\right] - H_{oz} \, a \cosh\eta
$$

$$
+ (H_{ox}\cos\psi + H_{oy}\sin\psi) \frac{a\cos\theta(\cosh^2\eta-1)^{\frac{1}{2}}}{(1-\cos^2\theta)^{\frac{1}{2}}} \Bigg\} \qquad \text{(E.19b)}
$$

$$
B_{\psi III} = \frac{-\mu_1}{a(\sinh\eta\,\sin\theta)} \Bigg\{ (-E'\sin\psi + F'\cos\psi)(1-\cos^2\theta)^{\frac{1}{2}}
$$

$$
\left[\frac{2\cosh\eta}{(\cosh^2\eta-1)^{\frac{1}{2}}} - (\cosh^2\eta-1)^{\frac{1}{2}} \ln\left(\frac{\cosh\eta+1}{\cosh\eta-1}\right)\right]
$$

$$
+ (H_{ox}\sin\ldots - H \ldots \text{[illegible]} \ldots \cos^2 \ldots)^{\frac{1}{2}} \Bigg\} \qquad \text{(E.19c)}
$$

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